COMPOSITION

Algebraic Composition Defined under Geometric Condition (Brown)

Let's start with something more mundane: 'what's it about 'coffee' and about 'cup' that allows one to 'add' coffee to cup?'

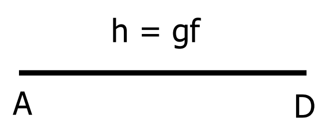
Before we answer the above question, let's go back to the notion of 'algebraic composition defined under geometric condition.' To make this notion little bit more concrete, consider 2 geometric objects, say, 2 line segments f and g, depicted as below:

There are many things about the above geometric objects that one could talk about, but there are two things about each of the two geometric objects f and g that are of particular relevance for our present purposes; they are the endpoints of each of the two line segments f and g. Let's denote the endpoints of f as 'A' and 'B', and the endpoints of g as 'C' and 'D' as shown below:

Now let's go back to where it all started i.e. composition. How do we compose, or add, or put-together the two objects f and g, or more explicitly the two line segments f and g? We can put f and g together by bringing them close to one another such that the end-point B of f coincides with the end-point C of g. In other words, we can put-together f and g if and only if B = C. The composite of f and g is a line-segment gf with A and D as end-points.

Composition of

is defined if and only if B = C, and the composite is



The most important thing to note is that when we want to compose two things, the two things to be composed must have something in common to form the composite.

Now let’s go back to coffee and cup and try to answer how we get to add coffee to cup using the condition for composition: something common! Since we all add coffee to cup all the time, there must be something in common between coffee and cup, which is volume. But for the fact that both coffee and cup have the common property of volume, we wouldn’t be able to add coffee to cup.

Functions, Rules, and Equality

Now let’s go back (I hope all this going back is not too uncomfortable) to functions. Consider 2 functions f: A → B and g: C → D. Now let’s find out under what conditions or what are the conditions that the above two functions 'f' and 'g' have to satisfy in order for us to form the composite of the two functions. With the geometric conditions fresh in mind, we could take a guess at it, but before we do that, let's get very clear about what a function is. The notation for a function is

$$f: A \to B$$

The function

$$f: A \to B$$

has 3 things:

1. a domain object 'A', a set
2. a codomain object 'B', a set
3. a rule

Let's dig little bit deeper into the 'rule' that's part of the definition of function.

rule:

for each element 'a' of set 'A'

there is exactly one element 'b' of set 'B'

such that 'b' is the value of the function 'f' at 'a'

$$f(a) = b$$

Any 'f' that satisfies the above property (labeled 'rule') is considered a function from a set 'A' to a set 'B'.

Before we get back to composition, there is one more thing that we need to get clear about function, which also happens to be about the 'rule'. Recall that a function has 3 things:

$f: A \rightarrow B$

1. a domain set A
2. a codomain set B
3. a rule (which assigns an element of the codomain to every element of the domain)

From this definition of function, there is much we can say; to begin:

Two functions 'f' and 'g' are different if:

1. they have different domains; e.g. $f: A \rightarrow B$, $g: C \rightarrow B$
2. they have different codomains; e.g. $f: A \rightarrow B$, $g: A \rightarrow C$
3. they have different domains & codomains; e.g. $f: A \rightarrow B$, $g: C \rightarrow D$

How about rules? Are two functions different if they have different rules? For this we go back to the definition of 'equality of functions'. Two functions 'f' and 'g' are equal if they have the same domain, the same codomain, and if and only if they have the same value for each and every argument. Given $f: A \rightarrow B$ and $g: A \rightarrow B$, function $f = g$ if and only if $f(a) = g(a)$ for all 'a' in the domain A. With this definition of equality at the front of our minds, let's try to answer the question: 'are two functions different if they have different rules?'

Let's consider two rules, wait, before we do that let's consider two functions with same domain and same codomain so that the only thing different is the rule corresponding to the function, so that we can clearly see the relation between 'rule' and 'function' in terms of 'same' and 'different'.

Consider two functions $f: N \rightarrow N$ and $g: N \rightarrow N$, where N is the set of numbers; $N = \{1, 2 \ldots\}$. Now let's consider two different rules, one for 'f' and another for 'g'. For 'f', let the rule be 'take the input 'n' and add 1 i.e. $(n + 1)$, and keep it aside for now, take the input and subtract 1 i.e. $(n – 1)$; now multiply them both i.e. $(n + 1)(n – 1)$; in terms of equations:

$$f(n) = (n + 1)(n - 1)$$

For 'g', let the rule be 'take the input n, square it i.e. (n^2), and subtract 1 i.e. $(n^2 - 1)$; in the format of equations:

$$g(n) = (n^2 - 1)$$

Now if we look at the two rules for 'f' and 'g', they are different: simplistically speaking, one (f) requires the operations of addition, subtraction, and multiplication, while the other (g) requires just multiplication and subtraction. Clearly we have two different rules for $f: N \rightarrow N$ and $g: N \rightarrow N$, but going by the definition of equality of functions,

$$f = g \text{ if and only if } \quad f(n) = g(n)$$

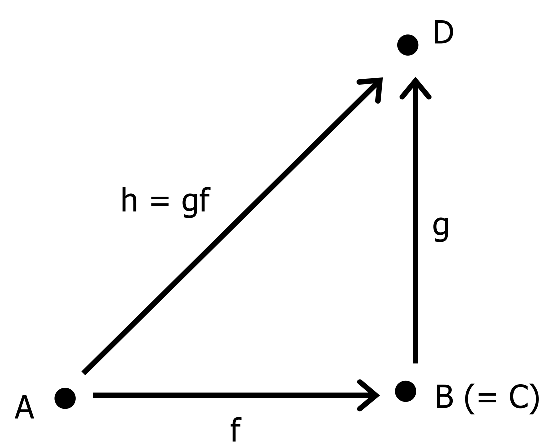
From the above

$$f(n) = (n + 1)(n - 1) = (n^2 - 1) = g(n)$$

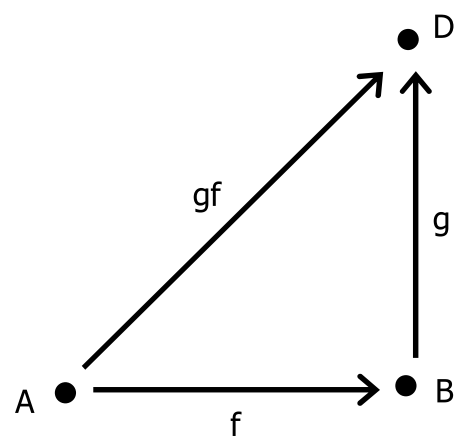
So, we find that even though the functions 'f' and 'g' have different rules, they satisfy the conditions for equality of functions. Thus we note that functions with different rules (but not different domains, or different codomains, or both i.e. different domains and different codomains) can be equal.

Composition and Associative Law

Now with the 'equality of functions' in place, let's return to 'composition of functions.' Consider two functions f: A → B and g: C → D. Taking a cue from 'algebraic composition defined under geometric conditions', we say the composite of two functions f: A → B and g: C → D is defined if and only if the codomain of the first function f is same as the domain of the second function g i.e. B = C. Once this condition in terms of domain and codomain of functions is satisfied so as to make the functions composable, we are now in a position to find the composite of 'f' and 'g'.



The composite of f: A → B and g: B → D written as 'gf' (and read 'g after f') has as domain the domain of 'f' i.e. 'A', and as codomain the codomain of 'g' i.e. ‘D’. Now looking at the above diagram we notice that there are two pathways to go from ‘A’ to ‘D’: 1. first go from ‘A’ to ‘B’ and then from ‘B’ to ‘D’; and 2. go from ‘A’ to ‘D’ directly. Looking back at the diagram, redrawn below



Now, first we can find the value of ‘f’ at, say, ‘a’ of ‘A’; f(a) = b, and then evaluate the function ‘g’ at ‘b’ i.e. g(b) = d or expanding we have

g(b) = d

g(f(a)) = d

since element ‘a’ of ‘A’ is a function from singleton set to ‘A’

g(fa) = d

Now let’s look at the second route gf: A → D. Evaluating (gf) at ‘a’, we get (gf)a = d. In order for the two paths to lead to the same destination, they have to evaluate to the same‘d’ of D for the same ‘a’ of A i.e.

g(fa) = d and (gf)a = d or

g(fa) = (gf)a

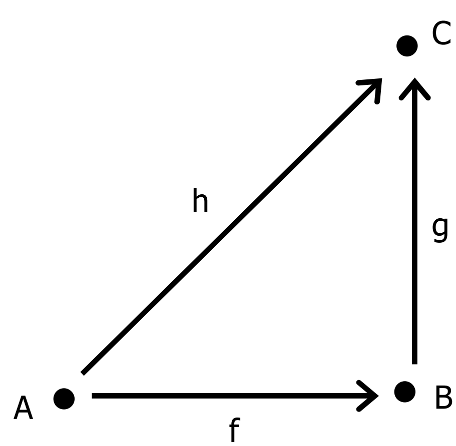
i.e. the above expression or identity or law has to be satisfied.

g(fa) = (gf)a = gfa

is nothing but the associative law. So, we may say composition of two functions is a special case of associative law.

Commutative Diagram

No, we are not quite done with our diagram (external diagram; since we are not looking at the innards i.e. elements of objects A, B …):



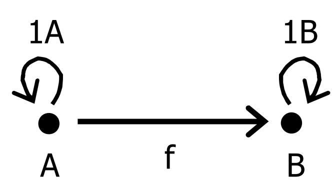
The above diagram, in which there are two paths from a point 'A' to another point 'C' and in which taking either one of the two paths is same i.e. taking 'gf' from A to B to C is same as (or equal to) taking the other path 'h' from A to C, is called commutative diagram. Making it crisp, if, in the above diagram, h = gf, then the diagram is said to be commutative.

Identity Laws

Given any function f: A → B we can readily construct two commutative diagrams along the following lines: First note that to each set A we can associate a function called identity function, which has the set A as both domain and codomain, and which assigns to each element 'a' of the domain A the same element 'a' of codomain A. Thus given a function f: A → B, we can construct two functions 1A: A → A and 1B: B → B.

Given the three functions f: A → B, 1A: A → A, and 1B: B → B, how many composites can we form?

Recollect that in order for a composite of 2 functions to be defined the codomain of the first function must be same as the domain of the second function. Looking back at the above 3 functions, we note that (i) the codomain of 1A is same as the domain of f, so we can form the composite f1A of 1A and f; and (ii) the codomain of f is same as the domain of 1B, so we can form the composite 1Bf of 1B and f. In terms of external diagrams:



or more explicitly

$$\begin{tikzcd}
& B \\
A \arrow[ur, "f1_A = f"] \arrow[r, "1_A"'] & A \arrow[u, "f"']
\end{tikzcd}$$

and

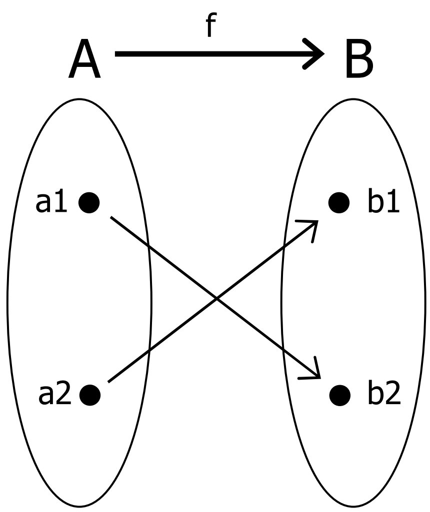
$$\begin{tikzcd}
& B \\
A \arrow[ur, "1_B f = f"] \arrow[r, "f"'] & B \arrow[u, "1_B"']
\end{tikzcd}$$

Now let's look at a simple illustration of the commutativity of the above two diagrams i.e. let's see an example of the following two identities:

(i) f1A = f

(ii) 1Bf = f

Let's look at the internal diagram of a simple function f: A → B

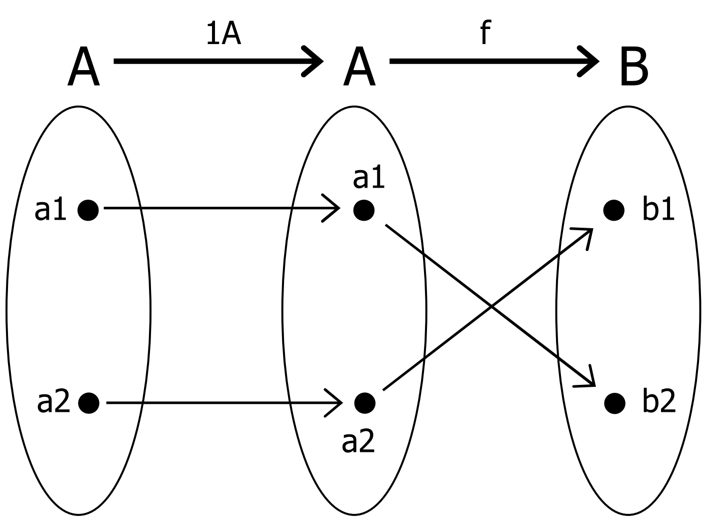


f(a1) = b2

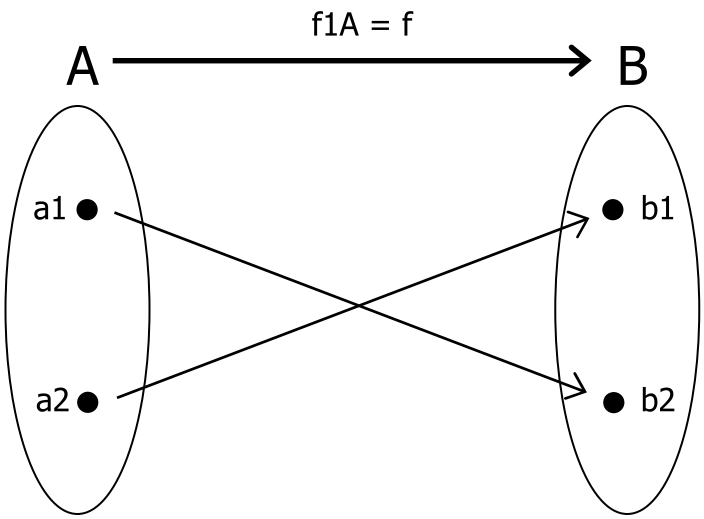
f(a2) = b1

To see that f1A = f

The composite of

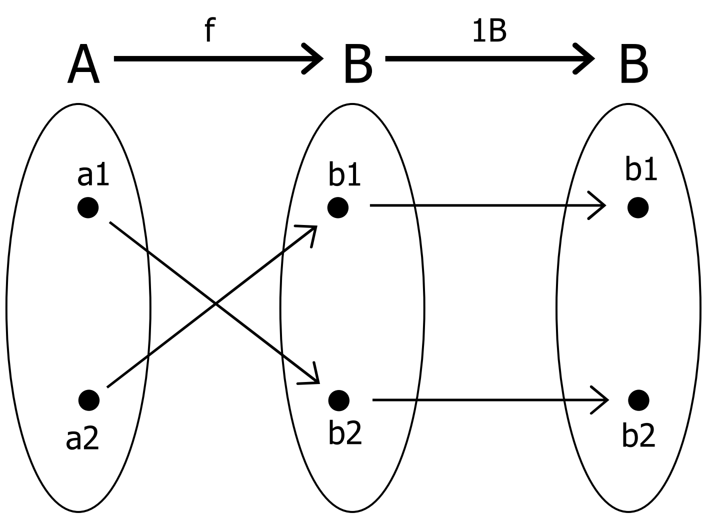


is

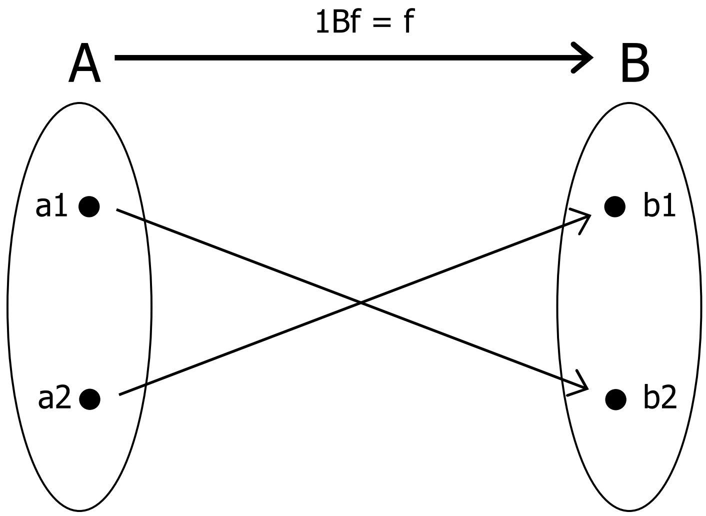


To see that 1Bf = f

The composite of



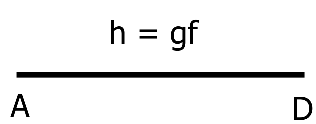
is



Composing Squares

Early on we began by noting ‘algebraic composition defined under geometric conditions’, which led to the condition that to compose 2 line segments, the target of the 1st line segment must be the same as the source of the 2nd line segment. Composition of

is defined if and only if B = C, and the composite is given by

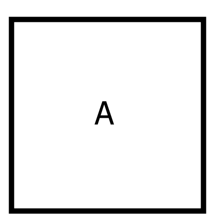


Now let's see if we can stretch the geometric intuition into additional algebraic concepts. Since we started with line-segments, which are 1-dimensional and found that the composition of 1-dimensional geometric line-segments or algebraic functions is not always defined; it is defined if and only if the 0-dimensional source of the 2nd line-segment (function) is the (0-dimensional) target of the 1st line-segment (function). So far so good or so well.

Now that we are at 1-dimensional line-land, we can go either 0-dimensional point-land or 2-dimensional flat-land (c.f. Abbot), and ask the same question of composition or putting together.

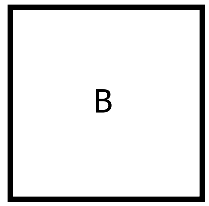
First, let's look at the case of 0-dimensional points. Composition or putting-together of structure-less points or elements, unlike the case of functions, where composition is not always defined for a pair of arbitrary functions, is always defined, which is nothing but the collection of elements into a set. In other words, in the case of 0-dimensional points or elements "composition" of elements into collections or sets is always defined.

Now, let's go in the other direction: from 1-dimensional line-segments to 2-dimensional squares. Consider a square A

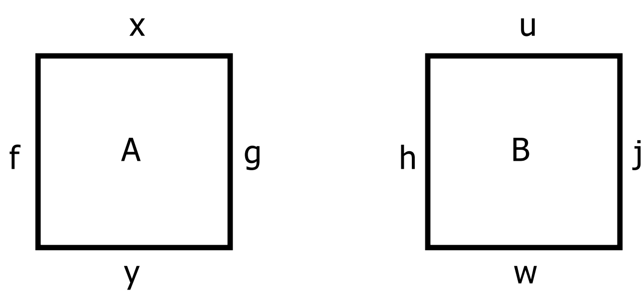


and note that the square A has (i) horizontal edges, and (ii) vertical edges.

Now consider another square B



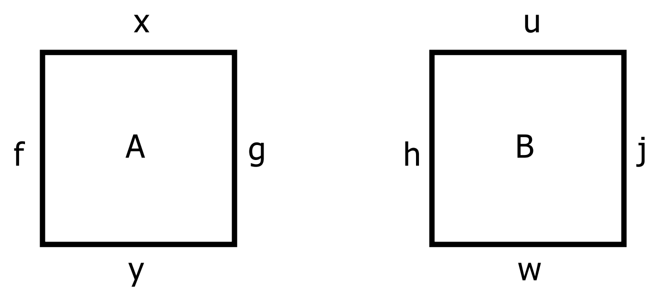
Now back to our good-old question of composition. Given 2 squares A and B, how and under what conditions can we compose them and what are the composites? Consider the two squares:



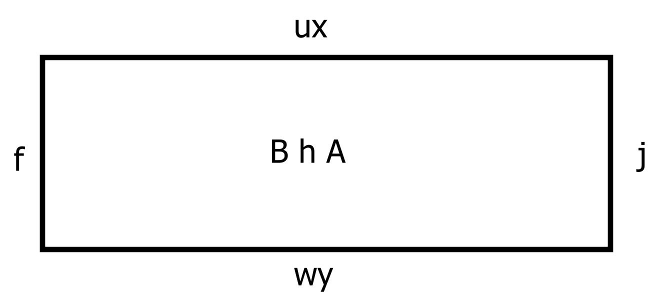
The square A has as vertical edges f and g, and as horizontal edges x and y. The square B has as vertical edges h and j, and as horizontal edges u and w.

Taking a cue or extending the reasoning employed in defining composite in 1-dimensional line-segments case, we try to find the conditions under which composition of squares is defined. Those of you who played Lego might have the answers. In any case we find that in the case of 2-dimensional squares, given two squares we can form (i) a composite by stacking the squares vertically or (ii) a composite by stacking the squares horizontally. Let’s now get specific or concrete.

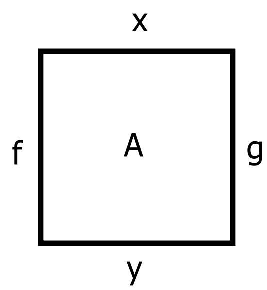
Given 2 squares A and B

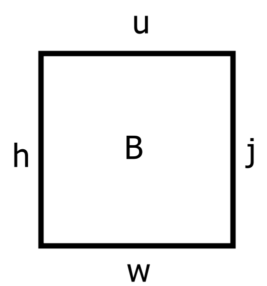


the horizontal composite B h A is defined if and only if the vertical edges g and h coincide, and the composite is:



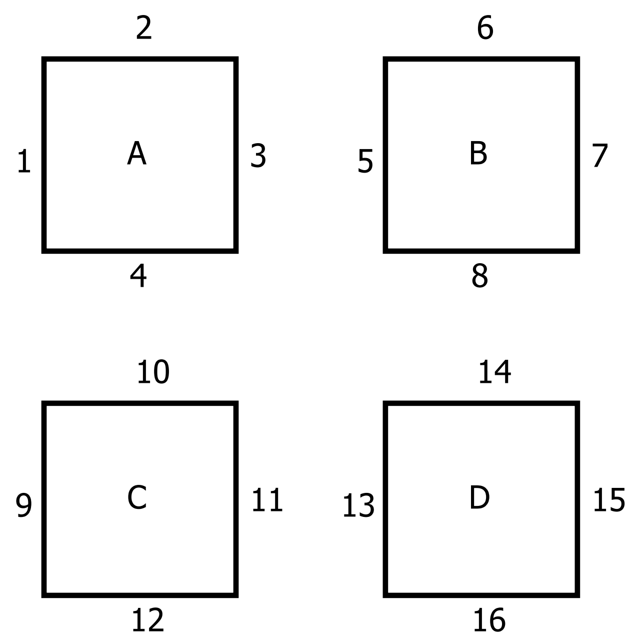
Similarly, given two squares





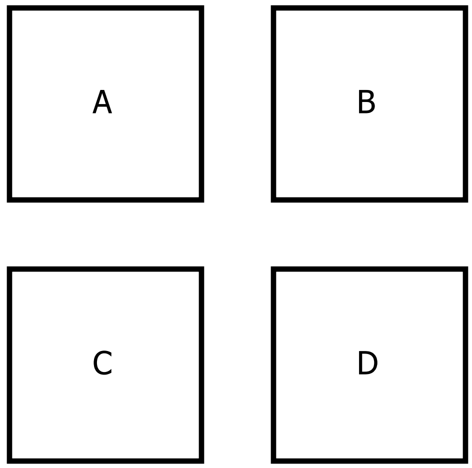
the vertical composite of A and B, B v A is defined if and only if the horizontal edge y of A coincides with the horizontal edge u of B, and the composite B v A is:

Now let's spice up squares a bit. Consider 4 squares A, B, C, and D, and the two operations v and h (vertical and horizontal composition, respectively):

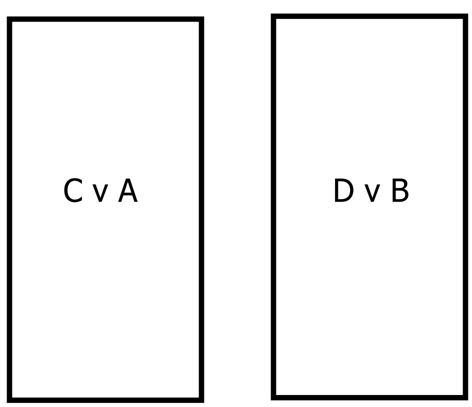


The squares are labeled with capital letters as usual, and the edges are labeled with numbers (used simply as distinct symbols). Looking at the 4 squares above we can think of forming the composite of 4 squares in 2 ways: (i) forming 2 vertical composites first and then forming the horizontal composite of the 2 vertical composites; and (ii) forming 2 horizontal composites first and then forming the vertical composite of the 2 horizontal composites.

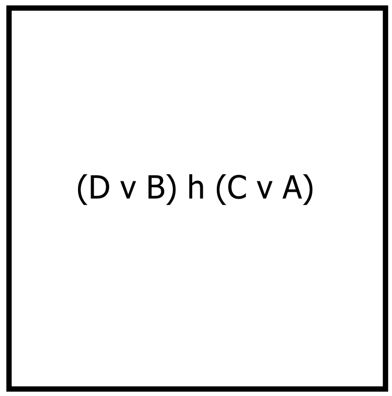
Pictorially, we can, given



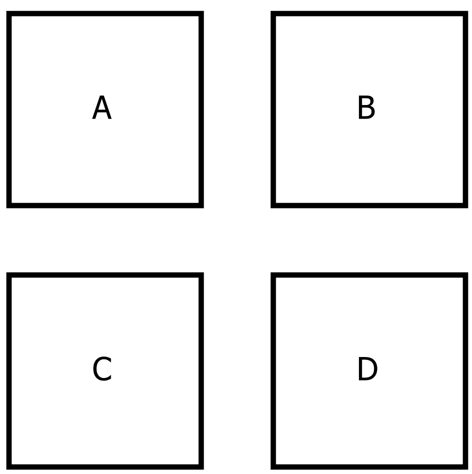
(i) first form 2 vertical composites (here and later on we assume that the conditions for vertical and horizontal compositions i.e. coincidence of the appropriate edges is satisfied or given):



and then form the horizontal composite of the above 2 vertical composites:



Redrawing the 4 squares again:



(ii) first form 2 horizontal composites (remember the conditions for compositions are assumed to be given or satisfied):

$$\boxed{B \ h \ A}$$

$$\boxed{D \ h \ C}$$

and then form the vertical composite of the above 2 horizontal composites:

$$\boxed{(D \ h \ C) \ v \ (B \ h \ A)}$$

Now to the obvious question: Given that, given 4 squares, (i) we can first form vertical composites and then form their horizontal composite; or (ii) we can first form horizontal composites and then form their vertical composite, are the two composites equal? In other words, does the following identity

(D v B) h (C v A) = (D h C) v (B h A)

hold true? Well, if it holds true, we say it—the interchange law—holds true; if not, we say the interchange law doesn't hold true in the given case.

Finally all this geometry, i.e. squares, edges, horizontal, and vertical composition is but to get to the notion of a function which takes functions to functions. Our familiar function takes elements (of a domain set) to elements (of a codomain set). If we think of a function f: A → B as a process that transforms an object A into an object B, it's not much of a leap of imagination to think of a process which transforms a process f: A → B into a process g: C → D as shown below:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, phantom, "\Rightarrow h"] & D \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g"'] \arrow[ur, phantom, "\Downarrow v"] & D \end{tikzcd}$$

These have the usual identities, associative laws, compositions, etc., but that's all for later.

Composing Commutative Diagrams

Before we close, let's go back to commutative diagrams. Consider 2 commutative diagrams CD1 and CD2:

$$
\begin{tikzcd}
C & \\
B \arrow[u, "g"] \arrow[ur, phantom, "\mathrm{CD1}"] & A \arrow[l, "f"] \arrow[ul, "h"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C & D \arrow[l, "q"'] \\
\arrow[ur, phantom, "\mathrm{CD2}"] & A \arrow[u, "p"'] \arrow[ul, "r"]
\end{tikzcd}
$$

with h = gf and r = qp. We can form the composite of the 2 commutative diagrams if h: A → C is equal to or coincides with r: A → C. Let's say it does i.e. h = r, so that we can form

$$
\begin{tikzcd}
C & D \arrow[l, "q"'] \\
B \arrow[u, "g"] & A \arrow[l, "f"] \arrow[u, "p"']
\end{tikzcd}
$$

and we note that there are 2 paths from A to C: 1. from A to B to C and 2. from A to D to C. The first path is the composite 'gf' which is equal to 'h' which in turn is equal to 'r' which in a final turn is equal to 'qp'. Written crisply,

$$
\begin{aligned}
gf &= h \\
&= r \\
&= qp
\end{aligned}
$$

Thus we find that a big diagram formed of smaller commutative diagrams is also commutative, or the composite of commutative diagrams is commutative. In closing, if it seems as though the presentation is too pedantic, it's simply to, in the words of Lawvere, "discern the germ of nontrivial in the trivial."