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Everyday physics of extended bodies or why functionals need analyzing

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Dedicated to Bernhard Banaschewski on the occasion of his 90th birthday

Abstract. Functionals were discovered and used by Volterra over a century ago in his study of the motions of viscous elastic materials and electromagnetic fields. The need to precisely account for the qualitative effects of the cohesion and shape of the domains of these functionals was the major impetus to the development of the branch of mathematics known as topology, and today large numbers of mathematicians still devote their work to a detailed technical analysis of functionals. Yet the concept needs to be understood by all people who want to fully participate in 21st century society. Through some explicit use of mathematical categories and their transformations, functionals can be treated in a way which is non-technical and yet permits considerable reliable development of thought. We show how a deformable body such as a storm cloud can be viewed as a kind of space in its own right, as can an interval of time such as an afternoon; the infinite-dimensional spaces of configurations of the body and of its states of motion are constructed, and the role of the infinitesimal law of its motion revealed. We take nilpotent infinitesimals as given, and follow Euler in defining real numbers as ratios of infinitesimals.

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1 Introduction

From the study of calculus and related mathematical sciences, one sometimes hears of a subject called functional analysis and perhaps even of the fact that thousands of mathematicians are devoting their lives to its study. Also in connection with attempts to describe the workings of computer programs one sometimes hears of functionals. Statistics too is largely about functionals. What are functionals, and why do they need so much analyzing? It would be desirable to have a basic grasp of such a fundamental concept in order to be able to approach in imagination the problems and solutions with which experts occupy themselves and which in fact impinge constantly on our everyday lives in this technological age.

I do not intend to give technical details here concerning problems and solutions, but I can indicate a general common feature. The experience of 300 or 400 years of modern science shows that in order to describe the motion of electromagnetic waves, of fluids and gases, and of responsive bodies such as airliners, and especially in order to be able to intervene in that motion we must deal in a precise way with dependencies which relate figures and quantities in spaces of an infinite number of dimensions. These dependencies take the form of functionals. Conversely, almost any dependency which is mathematically reasonable could possibly show up in nature for everyday materials such as glass or wood or silicon, or in extreme environments such as the cylinder of an auto engine, a lightning bolt or the heart of a mammal.

What I will describe in detail is a diagrammatic method for approaching such situations as they are modeled within conceptual frameworks known as cartesian closed categories of smooth spaces. Thus I need to explain what is commonly meant by **spaces** in general, **smoothness**, **categories**, and **cartesian closedness**; within such a framework, **functionals** have their natural place as the dependencies whose analysis depends on extensive study of particulars, guided however by the general features of the framework.

2 Spaces, cartesian Closed Categories and functionals

By spaces mathematicians have long meant not only ordinary three-dimensional physical space S, but a whole panoply of related objects; each of these can serve as the domain of variation for variable quantities, and simultaneously as an arena within which variation and motion can be imagined.

For example, a one-dimensional space T is often considered as a model of the time line, and even more basically, there is the very important space 1 consisting just of one point. In addition to these, there are important operations such as **product**, **sum**, and **exponentiation** (which I will describe) for producing more complicated spaces out of given ones. It is due to the condition, that the results of those operations are again spaces of the same category, that a category of spaces is called "cartesian-closed". Crucial to all thinking about spaces is the conceptualization of **maps** between them. For example, a map $T \to S$ can represent a particular path of a point particle in space through time, and more generally, a map $T \to X$ can represent a particular process through time in space X of placements or states of an extended body, as explained below. A map $1 \to T$ signifies a particular choice of a point in time, and a map $1 \to S$ a particular choice of a point in space S; in general, a map $1 \to X$ is usually called simply a **point** of X. The particular nature of 1 itself is determined by the fact that for any general space X there is just one map $X \to 1$.

The most basic operation on maps is composition, which to any pair of maps f, g for which the target space of f is the same as the source space of g, yields a new map gf whose source is that of f and whose target is that of g. As a special example, composing a path with the choice of a point in time gives as a result the point of space where the path is passing at that time.

Since composition is associative, one has that composition satisfies the familiar "function of a function" rule

$$(gf)t = g(ft)$$

for all points in the source space of f. However, this associativity equation holds for any map t (which is not necessarily a point) and moreover for liberating our thinking it is crucial to recognize that in typical useful categories, a map f is not determined by the ensemble of its "values" ft for points t of its source space; we may have to evaluate f at more general elements of its source space in order to fully determine its particularity. To avoid confusion, it is better not to use the expression "function of a function" for composition, since that expression better describes the much more general idea of functional, as was pointed out in 1887 by Volterra when he introduced the concept (though not the word). Before we arrive at the definition of that concept, we need to make explicit a few more elementary properties of a category of smooth spaces. Among the endo-maps of a space X (that is, maps whose source and target are both the same X) there is in particular the identity map 1_X which satisfies the expected equations with respect to both fore and aft composition. For some maps $X \to Y$ there exist maps $Y \to X$ with composite 1_Y; any such map is called a "section" of the original map and a good deal of mathematical striving consists of seeking and classifying sections. However, if a section itself has a section then there is only one and the original map is called an isomorphism; two spaces between which there exists at least one isomorphism have all significant properties in common and we may write $X \cong Y$ to signify that X and Y are "isomorphic" in this sense, although it is important in most contexts to consider whether such isomorphisms are "natural" in a sense which category theory was invented to make precise.

The important operation of "cartesian" **product** can be applied within a category of smooth spaces: For any two spaces A and B there is a third space $A \times B$ equipped with special maps to A and B respectively, called projection maps, which satisfy the so-called universal property that for any space Q and any pair of maps $Q \to A$ and $Q \to B$ there is exactly one map $Q \to A \times B$ whose composites with the projections express the given pair. Thus (taking $Q = 1$) the points of a product $A \times B$ uniquely represent the ordered pairs of points, and even more importantly, (taking $Q = T$) a process in $A \times B$ represents an arbitrary parallel pair of processes, one in A and the other in B. Sometimes attempts are made to define a map to somehow consist of ordered pairs; that approach is not satisfactory in general, since we must consider the idea of map as a primitive notion, but the attempt is based on the fact that for any map $A \to B$ there is as a special case of the universal mapping property, another map $A \to A \times B$ called its graph which is a section of the projection to A and whose composite with the projection to B recovers the given map.

It is ordinarily assumed that there is an isomorphism of spaces

$$S \cong T \times T \times T$$

wherein to each of the three projections we imagine a section which is a path called a "coordinate axis"; such an isomorphism is not "natural", because there is no preferred choice of such a "coordinate system".

Although a map whose target is a product can be trivially analyzed into maps whose targets are the factors, such is by no means true for maps whose source is a product. Such maps $A \times B \to C$ are called binary operations and include (for example if $A = B = C$) the rich structures known as Lie groups $G \times G \to G$. In general the output of a binary operation depends with non-trivial particularity on the interaction of the components of an input element.

The operation of **sum** when applied to two spaces yields another space equipped with injection maps to it from those summands, satisfying the universal property "opposite" to that of product projections. In particular, for any map $A \to B$ there is a cograph $A + B \to B$ for which the injection is a section. When we try to internally picture the action of a particular map, or define a map via a table of values, or to picture composition, it is often the cograph which we actually use.

Translating the cograph description into the graph picture of a given map is an important elementary exercise in becoming intimately acquainted with the particularities of that map. Because our category of smooth spaces is a very non-linear category, the distributive law relating sums and products is actually true in the sense that the obvious map

$$A \times X + A \times Y \to A \times (X + Y)$$

is always an isomorphism. That fact will follow from the existence of the key operation of **exponentiation**, or map space construction, to which we now turn.

For any two spaces W and A, it is very important that there be a third space W^A, called a map space, with the universal property that for any space X, any binary operation $A \times X \to W$ has associated to it a well-defined map $X \to W^A$, and this association is invertible. This association is mediated by a special map $A \times W^A \to W$, called an evaluation map, similarly to the mediation of product structure by the projection maps. By considering the case $X = 1$, we see that the points of W^A uniquely represent the maps $A \to W$; hence the name map space, but again more general elements than points are needed to detect the "inner cohesion" that any space, in particular a map space, has. Now, finally we can define the notion of **functional**, generalizing in a natural way Volterra's conception: It signifies any map

$$W^A \to Y$$

whose source space is equipped with a structure of a map space. Sometimes the word "operator" is used to signify a more structured situation in which the target space is also a map space. Let us briefly survey some of the many forms and contents of functionals: It can easily be shown that $W^{A+B} \cong W^A \times W^B$ so that defining $2 = 1 + 1$ we can consider that a binary operation of the kind $W^2 \to W$ is a very special kind of functional; usually the term is only used in case the exponent on the source is a more extended space than 2. lf we are given a map $A \to B$, there results from the definition an induced operator $W^B \to W^A$ which is sometimes called a restriction operator; the action of the latter on points corresponds just to simple composition $A \to B \to W$, but since it operates also on more general elements of W^B this induced operator, like any functional, also preserves the cohesion or smoothness inherent in every object of our category, as will be made more precise below. The inducing process just described is often denoted (for fixed W) by $(\ )^*$ and is referred to as the **contra**variant functorality of the map space construction because it reverses composition.

Sections of restriction operators are often solution operators for "boundary value problems". Brouwer's fixed point theorem shows that such solution operators are usually not trivially induced by any retraction of B onto its boundary A. The solution operator s is always induced by a so-called Green function or Poisson kernel $\hat{\hat{s}}$ whose target is not A, but a "double dual of A".

If f is a boundary inclusion and if s is a solution operator for some PDE, then twice applying the basic transformation, the Green or Poisson operator is obtained as follows:

$$
\begin{tikzcd}
W^A \arrow[r, dashed, shift left, "s"] & W^B \arrow[l, shift left, "f^*"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \times W^A \arrow[r, dashed, "\hat{s}"] & W
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r, dashed, "\hat{\hat{s}}"] & W^{W^A}
\end{tikzcd}
$$

There is also covariant functorality for fixed exponent and variable base, as is easily seen. A final consequence of general formal nature that we mention is the canonical double dualization map or "Dirac delta" $A \to W^{W^A}$ which shows that any point of any space can be represented by a functional of a very special sort. But apart from these formal transformations and their ramifications there is no general way of getting all functionals. Certainly the ones of specific interest are not tautologically transformable into anything simpler, and therefore must be analyzed in their particularity as they arise in concrete situations.

3 Motion of bodies

For example, it is usual to model a body such as a cloud C as a space in its own right, usually with additional structure. A map $C \to S$ is then considered as a particular placement of the body in ordinary space and the space of placements S^C takes its place alongside the space S^T of paths as a fundamental example of an infinite dimensional space. A motion of the cloud during a time period T can be considered as a particular map $T \to S^C$, that is, a path in placement space. Because of the commutativity of products $T \times C \cong C \times T$, the motion can equally well be described as a map $C \to S^T$, which could be called a "placement in path space". Both of these representations of the motion are necessary in order to compute via composition the effects it has on some non-tautological functionals. For example, on placement space there is the important functional "center of mass" which is computed with the help of integration with respect to the mass distribution over the body; composing it with the motion in the first description gives the path in ordinary space which is traced by the center of mass of the body.

On the other hand, on the space of paths in S is defined the important operator of time differentiation $S^T \to V^T$, where V is a vector space of the same dimension as S; composing this with the motion in the second representation yields the map whose transpose describes the time variation of the velocity field on C itself. It should be clear that there are many similar examples whose computation must be approached with these concepts in mind to avoid confusion.

4 Smoothness and derivatives

I would now like to describe an additional particular feature which can be realized in suitable categories and which will justify calling them "smooth" by providing a concrete approach to the particular functionals of the kinds known as differentiation and integration.

This feature consists of a particular space I (for time instant) which has only one point $1 \to I$ (called zero). Although quite small (called by some "amazingly tiny") I is nonetheless not isomorphic to 1. A map $I \to X$ will be considered as an infinitesimal process or path and the map space X^I will play the role of the tangent bundle of X, which is indeed some sort of bundle over X since the point zero induces a restriction map $X^I \to X$. The chain rule of differentiation is essentially the covariant functorality of the map space construction applied to this case of the fixed exponent I. The basic relation between the instant and the time line is an algebraic operation $I \times T \to T$ which we consider as "addition" or infinitesimal time shift and which reduces to 1_T at zero; on the other hand, at any given point of time, the addition reduces to the inclusion $I \to T$ of the particular instant centered at that point in time.

First let us consider some properties and results which we need to make explicit concerning I itself. As for any endomap space A^A, we have a monoid structure (associative binary operation) induced by composition itself on the endomap space of I. This multiplication restricts to the subspace

$$\mathbb{R} \to I^I$$

of that endomap monoid defined by the equation expressing the idea of preserving the center point zero of the instant. This monoid of "speed-ups of the instant" is itself no longer infinitesimal and indeed is, I believe, a natural model of the system $\mathbb{R}$ of real quantities. This is a precise modern expression of Euler's characterization of real quantities as ratios of infinitesimals.

The monoid $\mathbb{R}$ acts naturally on any tangent bundle as velocity retardation or homotheties.

The precise measure of the fact that the instant is not a point is expressed by the axiom

$$\mathbb{R}^I \cong \mathbb{R}^2$$

which has been extensively studied in research papers and textbooks over the past fifty years. There is actually an isomorphism (of the type expressed by the axiom) for each choice of a unit of time; one of the implied projections to $\mathbb{R}$ is the bundle projection induced by the zero of the instant, but the other one expresses the idea that an infinitesimal path in $\mathbb{R}$ then (given the choice of unit of time) has a uniquely defined speed or slope.

In general, a map $\phi : I \times X \to X$ which reduces to 1_X at zero is called a vector field in X or first-order ordinary differential equation (ODE); usually that concept is defined to mean a section of the bundle projection from the tangent bundle of X, but since in our category the total tangent space is a map space, the universal property shows that this simpler formulation as a binary operation is equivalent: for each point of X, the vector field yields an infinitesimal path in X centered at the point, and that is done in a smooth manner.

Clearly a vector field is expressing a particular sort of "urge to move" in X; precisely, we define a solution process for such a given ODE to be a map $x : T \to X$ in the richer category of spaces equipped with vector fields, that is the composites $x(+)$ and $\phi(I \times x)$ are required to be equal along a solution x of ϕ. In this context the underlying space is often referred to as the state space for the processes which solve the ODE, and the pair consisting of the state space and the vector field may also be referred to as an infinitesimal dynamical system.

5 Laws of motion

A dynamical system in the sense of engineering physics, however, is an object significantly more structured than merely a vector field, because as Galileo discovered, it is appropriate to consider that states are already states of motion, and that the law of motion operates as a vector field of a particular kind on those. In fact, one of the first examples of a functional involved the hereditary dependence of the stress in certain materials upon a body's entire history of deformations. But here I will limit myself to the situation in which states of motion are adequately represented as infinitesimal histories of placements, in other words, $X = P^I$, the space of infinitesimal paths in the space P of placements such as $P = S^C$, where C is a body such as a cloud. Thus

$$X = P^I = (S^C)^I \cong (S^I)^C \cong (S \times V)^C \cong S^C \times V^C \cong P \times V^C,$$

since a natural elaboration of the basic axiom shows that the tangent bundle of ordinary space also "trivializes" to consist of pairs in which the second components are translation vectors, and hence we can say that the states of such a body C are pairs consisting of placements and velocity fields on it. A particular law of motion $I \times X \to X$ is then uniquely analyzed as an "urge to move" responding to the current placement and the current velocity field of the whole body (and satisfying a second equation, made possible by the double occurrence of I, which says that the second component is a derivative of the first). With help of the homothetical action of $\mathbb{R}$ on $X = P^I$ one can distinguish laws of motion which depend only on the velocity (commonly called frictional or viscous response) or which depend only on placement (commonly called elastic response) and even analyze more general laws of motion into combinations of such special laws with purely inertial ones. The total law of motion is constitutive for the particular material that the body C is made of and for the environment in which it finds itself.

If C does not degenerate to a finite discrete space of bare particles, the dependence of the response on placement can involve infinitesimal aspects, and indeed by requiring that this be the only dependence we can express the idea that the immediate urge to move for a particular portion of the body is a response to the placement and velocity field only of its immediate neighborhood (PDE's). An "inertial" law of motion which is fully homogeneous is called an affine connection and the projections into P of its solutions are called geodesics.

It is, for example, along laws of motion that variable quantities may vary. Variable quantities themselves are of two basic types, on the one hand, **intensive** quantities such as temperature, density, and pressure which on a body C belong to a space such as $\mathbb{R}^C$ and hence transform contravariantly along maps $C' \to C$ and, on the other hand, **extensive** quantities, such as volume, mass, energy and entropy, which act homogeneously on intensive quantities as distribution functionals $\mathbb{R}^C \to \mathbb{R}$ and so belong to the spaces

$$Hom(\mathbb{R}^C, \mathbb{R})$$

which depend covariantly on C. Both the intensive and extensive spaces of quantities are examples of the linear spaces that play a central role in functional analysis.

To take account of the fact that heat causes motion and motion causes heat, often the notion of state needs to be extended by augmenting the purely mechanical placement space P with thermal variables.

To sum up, I hope that I have been able to briefly indicate how the concept of functionals, interpreted in a smooth category with an infinitesimal instant I, is a powerful instrument in distinguishing various types of motions and variations of extended bodies and of significant quantities defined on them. I believe it is worthwhile to make the effort to learn the simple language associated with this concept, for it can act as a rigorous guide that does not have to be unlearned when approaching more precise formulations of particular problems and when planning the necessary steps in calculations that aim to solve these problems.

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References

[1] Bell, J.L., "A Primer of Infinitesimal Analysis", Cambridge University Press, Cambridge, 1998.

[2] Kock, A., "Synthetic Differential Geometry", Cambridge University Press (2nd Edition), Cambridge, 2006.

[3] Lavendhomme, R., "Basic Concepts of Synthetic Differential Geometry", Kluwer Academic Publishers Dordrecht, Kluwer Texts in Mathematical Sciences 13, 1996.

[4] Lawvere, F.W., *Toward the description in a smooth topos of the dynamically possible motions and deformations of a continuous body*, Cah. Topol. Géom. Différ. Catég. XXI (1980), 337-392.

[5] Lawvere, F.W., *Comments on the development of topos theory*, in the book: "Development of Mathematics 1950-2000", J.P. Pier (Ed) Birkhäuser Verlag, Basel, (2000), 715-734. See also in TAC Reprints 24 (2014), 1-22 (with author commentary).

[6] Moerdijk, I. and Reyes, G., "Models for Smooth Infinitesimal Analysis", Springer-Verlag, New York, 1991.

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