What is a Space?

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What is a Space?

A space is just an object in a category of spaces!

The implied further question is given a very general answer involving lextensivity, as well as a much more structured answer involving dialectically coupled (cohesive / discrete) pairs of toposes. Examples can be analyzed and constructed using the simple geometric paradigm of figures and incidence relations, by which any lextensive category can be embedded in a Grothendieck topos; more refined subtoposes of the latter are specified by Grothendieck coverings that embody the geometrical equivalent of existential / disjunctive conditions on these extended spaces: a specific example involves a generalization of Maschke means. The extended spaces always include the Hurewicz exponential spaces, for example, spaces of functions and distributions equipped automatically with the ambient sort of cohesion. Examples important for smooth, analytic, and algebraic geometry are infinitesimally generated, pursuing an observation that goes back to Euler. A smooth account of points and components for algebraic geometry over a non-algebraically closed field is achieved by replacing Cantorian abstract sets with Galois-Barr topos as the discrete aspect. The basic goal is to help make the advances in Algebraic Geometry during the past 50 years more accessible to students and to colleagues in related fields by utilizing the simplifying advances in categorical foundations during the same 50 years, especially guided by proposals made by Grothendieck in 1973.

Mathematics requires the general notion of a 'cohesively extended object'. The inside of such a cohesively extended object may encompass figures of various shapes. It can serve as domain of variation for intensively variable quantities such as functions, differential forms, cohomology classes, and also as domain of variation for extensively variable quantities such as measures, distributions, and homology classes. Moreover, structures like groups are typically not purely abstract, but rather move in such a space as cohesive background. I chose 'cohesive' as the general adjective because the historically and philosophically natural term 'continuous' has for over a century been restricted to one specific mathematical determination.

However, various determinations are needed in mathematical practice, so that a more concrete description involves several particular generalities such as smooth, analytic, algebraic, simplicial, cubical, et cetera. Actually, all those and more have non-trivial features in common, as partly described in my 2007 TAC article 'Axiomatic Cohesion'. The axioms given there apply to all of the categories just mentioned; but not included is the case of 'classical continuity' that had become the default model of cohesion. That default model, the category of 'topological spaces', has severe drawbacks. For example, Peano's space-filling curves should have been the signal to develop something like Grothendieck's 'tame topology', but instead the philosophical climate of the time led to their enshrinement as proof of the unfathomable nature of space that students of analysis would have to face by abandoning their geometric intuition (as Peano urged). In the 1940's Hurewicz made explicit a second defect and boldly replaced the default category with the k-spaces (later studied by Kelley, Steenrod, and Brown); his modification explicitly incorporated the cohesion that also infinite-dimensional spaces of variable quantities enjoy. Such cohesion has been implicitly required for Bernoulli's calculus of variations and for Volterra's analytic functionals.

For many reasons, the default is difficult to shake off. Thus, for example, we find today in Wikipedia the claims that both schemes and toposes are 'generalized topological spaces'; neither of those claims stands up under serious scrutiny. Such imprecise analogies do not honor the many insights discovered by Hausdorff and his successors and predecessors.

1

Because mathematically a space is internally characterized and transformed through closely related spaces, the best modern answer to our question is: 'A space is an object in a category of spaces'. Of course, that leads straight to another question, to which my axiomatic cohesion is a special answer. However, a very general answer, which includes indeed classical topology, schemes, and toposes as well, yet still has nontrivial consequences, is the notion of extensive category: that is a category C with coproducts, denoted +, such that for any two objects X, Y, the extended sum

$$C/X \times C/Y \to C/X{+}Y$$

is an equivalence. In other words, any space E spread over a sum X+Y splits itself uniquely as a sum E = E(X) + E(Y) of two pieces spread respectively over the summands. Intuitively, $E \to X$ is a kind of objective distribution over X, and weighing or counting E would give rise to a quantitative distribution or extensive quantity. (For example, we could consider a gas E in a box X as studied by Avagadro, or a tribe E in a continent X). The covariant functoriality of extensive quantities is achieved trivially by composing, and this process is additive (for example, the total or integral of a distribution over X is obtained by applying this functorality to X → 1). The pieces of E determined by distributing it in a sum are provably obtained by the special pullbacks along the injections. If an extensive category has general fibered products, or just products, these provably distribute over sums; for short we refer to such categories as lextensive or prextensive, respectively.

The prextensive condition on a category is a rudimentary general property of cohesion because, dialectically, it permits the detection of non-cohesion, for example by maps X → 2 where, by distributivity, 2 = 1 + 1 is a Boolean algebra.

2

The more subtle cohesion in the inside of any object X is typically detected by its geometry of figures and incidence relations. In interesting cases this analysis of the inside of X is adequately done with the help of a smaller category B → C and is just the slice category B/X; the objects of that category are the B-shaped figures in X and its maps (commutative triangles in C) determine the incidence relations (such as belonging, intersecting and so on). A subspace of X is characterized (Grothendieck 1958) by the fact that any figure can belong to it in at most one way. If an inclusion of categories B → C is adequate (Isbell 1960) then for any inclusion U → V between subspaces of any X, the condition that every B-figure in X that belongs to V also belongs to U, is sufficient for equivalence $U \simeq V$ in C/X. The shape-labeling functor B/X → B is a discrete fibration, and the adequacy is equivalent to the requirement that any functor B/X → B/Y be induced by a unique actual map X → Y in C provided that it preserves the labeling. (Such a functor is intuitively a ‘continuous’ process in the sense that it preserves the incidence relations of figures without tearing them.) Discrete fibrations are of course equivalent to set-valued functors; the elementary fact that the objects of ‘any’ category can be analyzed as functors should not be used to frighten students. The structuralist thesis that spaces ‘are’ structures is thus tempered by experience: each choice of an adequate category B → C yields an analysis of the spaces in C as geometric structures of shape B. (Dually, a choice of a co-adequate T → C yields a contravariant analysis of C in the opposite of algebraic structures of co-shape T; a T-algebraic structure is a discrete op-fibration over T, whose maps act as algebraic operations, and functors between such op-fibrations are homomorphisms because they preserve the operations labeled by the maps in T).

Volterra in the 1880’s used the term ‘elements’ for what we have called ‘figures’. Later the term ‘element’ came to suggest some special properties that a figure may not possess. Grothendieck around 1960 popularized the term ‘point’ for general figures, which had a sort of brave new science-fiction air but is rather counter-intuitive. Everybody knows that points are a very special kind of figure. How special they are becomes an important foundational question; Leibniz and Galois had already contributed some insights.

3

In analyzing or synthesizing a particular category C of spaces, the choices B → C and T → C of basic figure-shapes and basic function types give a particular view of the dialectics between incidence relations of figures and algebraic operations on functions in each particular space. Experience with projective space (Liouville's theorem) suggests that often a rather special choice T → C of 'algebraic theory' may (although not co-adequate in all of C) be co-adequate for a small B that is in turn adequate in all of C. Even fixing a 'small' such choice of set-valued bimodule T → B, very large compatible categories C arise: not only quotients of the 'affine' B-spaces may be important, but especially the function spaces of Bernoulli-Volterra-Hurewicz are host to the functional analysis over any T → B and should be incorporated in C, in order to reveal and utilize the intrinsic cohesiveness of the variable intensive and extensive quantities. Besides adequacy, a further constraint on enlarging C is governed by Grothendieck topology, guaranteeing that the gluing of spaces in C agrees as much as possible with what can be calculated in terms of T → B itself. In case B is itself extensive, that agreement is easily achieved for the initial case of disjoint gluing, so that less tautological Grothendieck topologies involve mainly restricting C to make some of the epimorphisms (existential statements from a logical point of view) agree with those in B. The bimodule T → B is often just a full inclusion, for example, (the opposite) of polynomial k-algebras among finitely-generated k-algebras, and together with a choice of Grothendieck coverings, such a bimodule is data for an analysis / synthesis of a topos C. A topos is a category with subspace representability, which implies both extensivity and Hurewicz exponentiation. The extent to which subspace representors can be analyzed in terms of 'Hilbert schemes' still needs to be clarified.

Godement's book was a fundamental reference in 1960, but Gabriel's 1966 lectures in Oberwolfach revealed that the typical topos is not a generalized topological space, but rather a category of spaces. Actually, Grothendieck, in his 1960 seminar on constructing analytic spaces, had implicitly already demonstrated that, as had Eilenberg & Zilber in their 1950 introduction of simplicial sets. Insights from Gabriel were incorporated into my 1967 Chicago lectures on Categorical Dynamics. The intended dynamical applications have been slow to materialize (as outlined in my 1992 talk for the colleagues in Scienza della Costruzioni), but the axiomatic method in those lectures led to the elementary theory of toposes. The axiomatic theory of synthetic differential geometry that emerged includes algebraic geometry; it further reveals that smooth topology can also be treated by analogous methods, including nilpotent infinitesimals. That treatment emphasizes the elementary role of a concrete algebraic theory that is not given by a presentation.

The developments of 1966-1973 deepened the conviction that the foundations of algebraic geometry, in particular, could be considerably simplified relative to the 1957-1962 version. This was emphasized by a 1973 Buffalo colloquium talk by Grothendieck in which he urged abandonment of the previous definition of scheme with its awkward dependence on the notion of locally ringed space. Of course, the category of schemes itself plays a key role, but more streamlined and conceptual constructions of it can be found (and axiomatic descriptions of the category itself can support directly the geometric intuition, without necessarily referring to any particular construction). Although Grothendieck's 1973 Buffalo colloquium talk was never published, fragments of his proposal circulated in the community, usually under the slogan 'spaces are functors'. Of course that is less than half of the story: even having analyzed the domain and codomain categories for these functors, there remains the fact that neither the minimal (or affine) category of representable functors, nor the maximal category of all functors, has the geometrically correct colimits; a wise choice of Grothendieck topology is needed to specify a good intermediate category whose quotients and sums are good. Such an intermediate category is a Grothendieck topos; we need to utilize the functional analysis that flows from that.

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Indeed even finitary functional analysis reveals a fact implicitly known to Euler: the full category of spaces is actually infinitesimally generated. There are several senses of ‘generated’ for such toposes. In general every object is a colimit of the objects in a given site, but the affine site in turn consists of finite limits of lines: the lines therefore generate the whole topos by means of two Giraud operations. However, another operation is the exponentiation that exists in any topos, and the affine site results from applying that operation to still tinier spaces.

The paradigm of functors and figures applies even to the classical topology, at least in its Frechet-Johnstone form where the basic figures are convergent sequences. However, the more refined axiomatic cohesion does not apply to that example because, unlike in simplicial topology, the notion of connected component is not a simple left adjoint.

There are many examples, especially those involving differential calculus, that not only satisfy the axiomatic cohesion, but are infinitesimally generated in the suggested sense of colimits of finite limits of exponentials of infinitesimal objects. More precisely, many toposes of spaces have a special infinitesimal object D, which could be referred to as ‘il punto nella punta’ because it has only one rational point and yet its self-exponential D^D has two distinct points while being connected in a suitable sense. That monoid object contains (as a retract) the space of those endomorphisms of D that preserve the point, and that space is just the multiplicative monoid (that contains the one-dimensional multiplicative group as the automorphisms of D). Since all finitary affine schemes are finite limits of copies of this line, it follows that the object D infinitesimally generates the whole topos. The objects D (and their closest relatives, such as D^2/2! are roughly the same in the various toposes of interest; only with a second iteration of the exponentiation do the distinctions between smooth, analytic, and algebraic become evident!

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The special properties of the infinitesimal objects like D, in relation to the ambient topos they generate, are very striking. They are typically among the objects forming a quintessential quotient topos in Johnstone’s sense, namely spaces that (suggestive of the monads of Leibniz) have the property that every connected component contains exactly one point. But even more potentially powerful is the existence of further right adjoints to the exponential tangent-bundle functors ( )^D; these in turn permit construction of Eilenberg-MacLane spaces for deRham cohomology.

In the case of algebraic geometry over a field k, the infinitesimal spaces will typically have finite-dimensional function algebras. A crude consequence of more precise statements by Hyman Bass concerning the ubiquity of Gorenstein rings, or by Birkhoff in the case of finitely-generated sub-directly irreducible algebras, is the principle that, in order to verify the consequences of a system of polynomial equations, it suffices to consider all their interpretations in finite-dimensional vector spaces over k. In other words, the infinitesimal skeleton D(X) → X of any space X is perceived as epimorphic by all affine spaces.

In accord with Grothendieck's philosophy of relativization, the treatment of cohesion is a contrast with 'non-cohesion' via a pair of toposes connected by four adjoint functors. The lower or relatively discrete topos may have very special properties, such as Boolean logic, without reducing fully to the Cantorian totally abstract sets. (Such reduction would in fact tend to destroy the good properties of the four connecting functors.) That lower topos supplies the distinction between points and the more general figures.

Since Galois, we have been accustomed to the idea that a solution to a polynomial equation over k 'exists' if it really exists in a finite extension of k. The significance of the principle 'Quantifiers and Sheaves' in my ICM address 40 years ago derived from the elementary observation that the subjective rule of inference describing the existential quantification of a property along a map expresses precisely the objective adjointness of the operation of direct image along a map applied to a subobject of its domain. The slightly less elementary observation was the answer to the question: 'To what extent do figures belonging to the image of a map actually come from the domain of the map?' The answer was in principle already supplied by Volterra, Poincare, and deRham with theory of exactness, and is now seen to reside in the sheaf-theoretic employment of coverings, a main example of Barr's theory of exactness. (As a by-product this observation brought out more sharply the actual content of the existential quantifier of logic: even in simple cases it only signifies actual existence when epimorphisms split, i.e., when local sections can be glued in the background category where theories are modeled.)

From such considerations we see that, for algebraic geometry over a field k that is not algebraically closed, the category of abstract sets is not a good base topos. A topos defined by a site that involves finitely generated algebras over k is cohesive over the topos of sheaves on a site dual to a category of finite field extensions of k. If we take all maps of fields as coverings, then this base topos is Boolean (as shown by Barr).