WHAT IS THE LENGTH OF A POTATO?

An Introduction to Geometric Measure Theory

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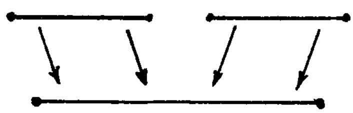
The question in the title probably sounds a bit peculiar; but I hope to persuade you that it has a unique sensible interpretation, and to show you several ways (at least for a potato shaped like a ball) to compute the answer. But my real goal is more ambitious: I hope to reform your intuition about geometry, to get you to incorporate into your picture of Euclidean geometry the sweeping changes in fundamental notions stemming from the work of Euler, Gauss, Riemann, Minkowski, and many others. For this reason I say very little about proofs (except to indicate where they can be found), and try to show the ideas in the simplest setting where they make their appearance.

Our topic is volume, area, length, and number. We begin with length. Imagine an idealized measuring stick, say of length one inch as pictured below. (I have drawn the heavy dots to emphasize that I'm thinking of a closed segment.)

Now this is really a rather poor instrument for measuring lengths. The defect is that if we magnify the segment by a factor of two, the resulting segment is not the disjoint union of two copies of the original; the two pieces have a one-point overlap.

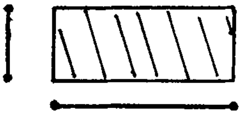


This suggests that our original segment was infinitesimally larger than one inch; its true size is 1 in + 1, the 1 for the one extra point. The basic lesson to be drawn from the geometers since Euclid is that it is not only possible, but even desirable, to keep track of this infinitesimal excess. So the "total size" of a solid figure in Euclidean space should not be a pure volume, but a formal sum of terms volume + area + length + number (so formally polynomials in the quantity in=inch). Let's calculate some examples:

1) A closed line segment of length L inches has size Lin + 1.

2) A closed rectangle has size

Win + 1

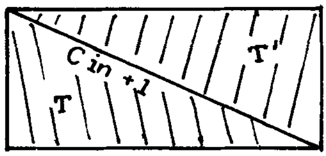


Lin + 1

$$(\text{Lin} + 1)\ (\text{Win} + 1) = \text{LWin}^2 + (\text{L} + \text{W})\text{in} + 1$$

3) A right triangle has its size computed as Euclid did, except that to take account of the excess

Bin + 1



Ain + 1

we must use

$$\text{Size}(T \cup T') = \text{size}T + \text{size}T' - \text{size}(T \cap T')$$

$$(\text{Ain} + 1)(\text{Bin} + 1) = 2\text{size}T - (\text{Cin} + 1)$$

$$\text{size}T = \frac{AB}{2}\text{in}^2 + \frac{A+B+C}{2}\text{in} + 1$$

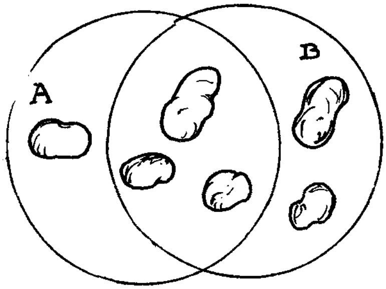
By now, perhaps you have begun to guess the significance of the terms in the size. The "area" is just the area as Euclid would have computed it. The "length" is one-half the perimeter. (One explanation for the factor ½: the boundary is only half exposed, so that for a two-dimensional creature to paint the exposed boundary requires only half as much paint as if he were to paint the one-dimensional figures which are the boundaries of our rectangle and triangle. These boundaries, as geometric figures in their own right, have their usual lengths.) The "number" of the figure is what came to be called the "Euler characteristic" after Euler's proofs that the number of a two-sphere is 2, and some investigations of one-dimensional figures.

Before going further, we must look a bit more closely at the number of a figure. Since the time of Euclid, there have been two great advances in our notion of cardinal number. From Cantor we learned to count infinite discrete sets, and from Euler we learned to count extended bodies. Of these two advances, Euler's has been by far the more important; but we seem, most of us, to have spent more effort retraining our intuitions to incorporate Cantor's ideas than Euler's.

Let's try to remedy that, at least a little, now. First an elementary observation about counting:

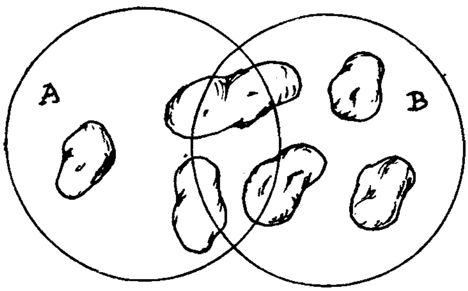
number$(A \cup B)$ = number (A) + number (B) - number $(A \cap B)$

as this pile of potatoes illustrates



$6 = 4 + 5 - 3$

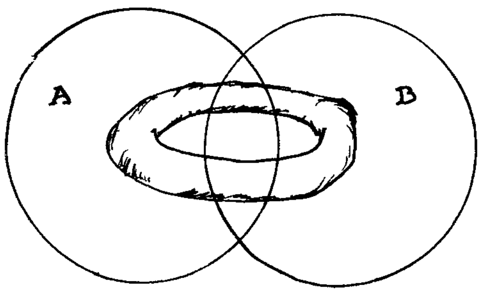
Of course any child could observe that, but how many of them have observed that the next example illustrates the same phenomenon?



$6 = 4 + 5 - 3$

Each object, be it a small potato or a large one or even a <u>piece</u> of a potato, counts as one.

We seem to get into trouble if our pile includes a doughnut:

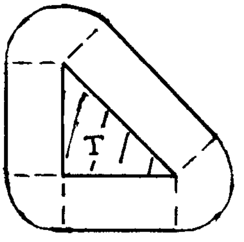


number $(A \cup B)$ = number (A) + number (B) - number $(A \cap B)$

$= 1 + 1 - 2 = 0$

So we're forced to count a doughnut as zero, if we want counting to be finitely additive when an extended body (or pile) is written as a union of parts which are not clopen. Of course, we're neglecting, for now, the important question of what sorts of bodies and what sorts of parts are to be allowed; what is apparent is that some sort of "combinatorial finiteness" is needed. To avoid these difficulties, let's restrict our attention for the moment to finite compact polyhedra, for which there is no difficulty in making precise the definitions of number, etc. and in proving the basic propositions. (But we reserve the right to draw examples from more general cases which have been worked out over the past century or so.)

We should illustrate at least one use of this refined notion of size, Steiner's formula. Even supposing one is interested only in the area of plane figures, one can ask for the area of the set of all points at distance at most R inches from a convex plane region T .



In the picture, it's clear that the large region decomposes as T $\cup$ (rectangles) $\cup$ (sectors of a disc), and the total area is Total area = $1 \cdot$ Area(T) + (2R in) Length(T) + ($\pi R^2 in^2$) Number(T), recalling that the length of T is one-half its perimeter. This is quite general, for any compact convex set in N dimensions, and is Steiner's formula. (The coefficients are just the n-dimensional measure of a ball of radius R in in n-space, here for n = 0, 1, 2.) The right side of Steiner's formula computes something even if T is not convex: one must think of the left side as the N-dimensional volume-integral of the function whose value at any point p is the number (=Euler characteristic) of the intersection of T with the closed ball of radius R in centered at p . Of course, when T is compact convex, this intersection is also, so this number is either 1 or 0, and the function becomes the characteristic function of the large region. This illustrates a general phenomenon in the whole subject: all problems reduce to the problem of correctly understanding number; length, area, etc. then are relatively easy to understand.

Another illustration of the primacy of number comes from the <u>integral-geometric interpretation</u> of length, etc. for, say, a figure in 3-space. To calculate the length (= 1-dimensional measure) look at all 2-planes in 3-space (because 2 = 3 - 1). Now on the space of planes there's a measure, unique up to a constant factor, invariant under rigid notions, giving open sets of planes positive measure and compact sets of planes finite measure; normalize it so that the set of planes meeting a line segment of length 1 inch has measure 1 inch. (It is best to think of this measure as valued in lengths, rather than pure numbers.) Now to calculate the length of our figure, simply integrate, over the space of planes, the function whose value at any plane is the "number of times the plane hits the figure", which must of course be interpreted as the Euler characteristic of the intersection of the plane with the figure. Similarly, to calculate the area of a figure, normalize the (area-valued) measure on the space of lines so that the set of lines meeting a square of side 1 in has measure 1 in^2, and proceed as before.

Returning now to the example which illustrated Steiner's formula, we can notice a bit more. The area, length, and number are not merely quantities associated to our triangle, but are the integrals, or total measures, of measures supported on the triangle: the area measure is the usual one; the length measure is one-half the usual length measure on the edges; and the number measure associates to the lower left vertex the measure ¼ and to each of the other vertices the measure 3/8, corresponding to the fractions of disc situated at each vertex in our picture. Federer has shown that for a class of subsets of Euclidean space called "sets of positive reach", significantly generalizing closed convex sets, one can use this idea to precisely define the measures, and to prove their relevant properties. For closed convex sets, Minkowski studied these quantities, which he called "Quermassintegralen". Unfortunately, he normalized them and indexed them in such a way as to obscure their geometric interpretation as length, area, etc.

The formula for the zero-dimensional measure is called the "Gauss-Bonnet formula", especially in the case of smooth manifolds with boundary. One example, simpler than our

number(isosceles right triangle) = ¼ + 3/8 + 3/8 = 1

is familiar to all children. To count the number of pieces of rope in a tangled mess of rope, it is unnecessary to separate the pieces; the number of the pile is concentrated at the ends of the pieces, each end counting one-half. It is a short step, conceptually, from this to the idea that the number of a solid object is also the integral of some local quantity; for a 3-manifold with boundary, for example a potato, in Euclidean 3-space, the Euler characteristic is the integral of a measure dm_o concentrated on the surface of the body:

$$dm_o = (4\pi)^{-1} R_1^{-1} R_2^{-1} \, ds,$$

where ds is the usual surface area measure, and R_1 and R_2 are the "principal radii of curvature" at a point. Note that since R_1 and R_2 have the dimensions of length, our measure has dimension $(\text{length})^{-2}\cdot$ area, so is a pure number, as it should be. This formula generalizes to give formulas for the k-dimensional measures dm_k for an n-manifold M with smooth boundary ∂M in n-space, for $k = 0,\ldots,n-1$;

$$dm_k = c_{n,k}\, p_{n-1-k}(R_1^{-1},\ldots,R_{n-1}^{-1})\; ds$$

where p_i is the i-th elementary symmetric function (homogeneous of degree i), ds is the usual (n-1)-dimensional surface area measure, and $c_{n,k}$ is a constant which can easily be computed, for example, by specializing to the case of a ball, for which we know m_k from Steiner's formula. For instance, for the potato, or solid body in 3-space,

$$dm_2 = \tfrac{1}{2}\, ds,$$
$$dm_1 = (2\pi)^{-1}(R_1^{-1} + R_2^{-1})\; ds,$$

dm_0 was calculated above, and dm_3 is the usual volume measure restricted to M . (It is a peculiarity of smooth figures that the lower-dimensional measures are spread all over the boundary. For polyhedra, dm_k is concentrated on the k-cells; but if you imagine approximating M by polyhedra you see why the measures get spread out.)

The observation that $m_0(S^n)$, the Euler characteristic of the n-sphere, is 2 if n is even, 0 if n is odd, generalizes.

$$dm_k(\partial M) = \begin{cases} 2\; dm_k(M) & \text{if } n-k \text{ is odd} \\ 0 & \text{if } n-k \text{ is even.} \end{cases}$$

Hence if we take n odd and $\partial M = \emptyset$, so a manifold without boundary, then not only is $\int dm_0 = m_0(M) = 0$, but in fact $dm_0(M)$ is <u>identically</u> zero, so $\int f dm_0(M) = 0$ for any integrable function f . More generally, for a manifold M without boundary, $dm_k(M) = 0$ in all odd codimensions, since it's $\tfrac{1}{2}\, dm_k(\partial M)$. Thus for instance for a 2-manifold with boundary (say in 3-space) the length measure is just one half the length measure on the boundary curves, and not spread over the surface; while the number measure is spread all over, like the area measure.

Let's look at one more example, to help visualize the measures: a solid cylinder M of radius R and height H . (This is topologically, though not smoothly, a manifold with boundary; so all of the preceding paragraph applies to it). $M = D\times I$, where D is a disk of radius R, and I an interval of length H . So the total measure is given by

$$m(M) = (m_2(D) + m_1(D) + m_o(D))(m_1(I) + m_o(I))$$
$$= (\pi R^2 + \pi R + 1)(H + 1)$$
$$= \pi R^2 H + (\pi R^2 + \pi RH) + (\pi R + H) + 1$$

where the homogeneous term of degree k gives $m_k(M)$. The formulas for m_3, m_2, and m_o are just the usual ones for volume, half of surface area, and Euler characteristic; but note the convenience of combining the terms into a single "polynomial", even for computing the surface area. More interesting is the analogous formula for dm ; for instance the length measure

$$dm_1(M) = dm_o(D) \times dm_1(I) + dm_1(D) \times dm_o(I).$$

Thus the length measure on a cylinder is the sum of two simpler (product) measures:

$dm_o(D) \times dm_1(I)$ is the product of the measure uniformly distributed over the circle ∂D, with total measure (the pure number) 1, multiplied by the length measure on the interval; so this term is uniformly distributed over the lateral surface of our cylinder, with total measure H (a length).

$dm_1(D) \times dm_o(I)$ is the product of the length measure on D, which is uniformly distributed over the circle ∂D, giving each arc a measure of ½ its length, multiplied by the pure number measure on I which gives each endpoint measure ½ ; so this term is concentrated on the top and bottom rims of our cylinder, and gives to each arc measure ¼ its length.

Notice that if we fix H and let R tend to zero, so that our cylinder tends to a line segment of length H, then the measures $dm_k(M)$ tend to those for the segment. This is an instance of a general continuity property of the measures, but the correct formulation of the appropriate notion of convergence of variable figures has been worked out only in special cases, for example for sets of positive reach by Federer. For compact convex sets, things are particularly simple, as Minkowski already knew: the sets are close if and only if they're close in the Hausdorff metric

$$d(A,B) = \sup\left(\left\{d(a,B),\ a \in A\right\} \cup \left\{d(b,A),\ b \in B\right\} \right).$$

Indeed, for (compact) convex sets A,B, the measures have many nice properties, for instance: $dm_k(A) \geq 0$; $A \subset B$ implies $m_k(A) \leq m_k(B)$.

Thus the length of B is greater than the length of A ; clearly this needn't hold if A and B are not convex, as a long spring in a small cylinder demonstrates; and $dm_k(A) \geqslant 0$ is false even for $k = 0$ for non-convex A . For non-convex A , there's a version of positivity: $m(A) \geqslant 0$, not coefficientwise as for convex A , but only in the ordering in which the term of highest degree dominates. For convex A , a lot is known about the possible values of the vector $(1=m_o(A), m_1(A), \ldots, m_n(A))$; the isoperimetric inequality is one constraint, and others are known, but I do not believe that a complete description of the image of this vector as A ranges over compact convex sets is known, even for $n = 3$.

To realize the utility of having the measures dm_k , instead of just the total $m_k = \int dm_k$, consider Pappus' formulas for the volume and surface area of a solid of revolution. These, you will recall, say that if we revolve the plane figure K (in the upper half-plane) about the x-axis to give a solid $\tilde{K}$, then

$$m_3(\tilde{K}) = m_2(K)\cdot 2\pi\, y_2 \quad \text{and}$$

$$m_2(\tilde{K}) = m_1(K)\cdot 2\pi\, y_1 \quad ,$$

where y_k is the result of averaging the distance y of a point from the x-axis, with respect to the measure $dm_k(K)$. (Of course, the second formula is usually multiplied by 2 , then saying that the surface area $2m_2(\tilde{K})$ is the perimeter $2m_1(K)$ times the average over the boundary curve of the y-coordinate times 2π .) But the important fact to notice is that y_k cannot be computed from m_k ; we need to know dm_k to do the averaging. Putting Pappus' theorems in this form immediately suggests another theorem:

$$m_1(\tilde{K}) = m_o(K)\cdot 2\pi\, y_o$$

This is also true, unless K meets the axis of revolution in a set of positive length $L = m_1(K \cap \text{axis})$; then L must be added to the right side. For one dimension lower, the main term disappears, but the correction does not:

$$m_o(\tilde{K}) = m_o(K \cap \text{axis}),$$

so the general form is

$$m_k(\tilde{K}) = m_{k-1}(K)\cdot 2\pi\, y_{k-1} + m_k(K \cap \text{axis}).$$

It's now easy to use these formulas to get alternative calculations of m_k for ball and cylinder, as well as for cones and so on.

All this has been greatly generalized, but much remains to be done. We have not emphasized the fact that a figure K has its own intrinsic (geodesic) metric $d_K(x,y)$, in terms of which the measures dm_k should have an invariant description, which I don't know how to give in a nice way. Closely connected with this are two other problems: what characterizes the metric spaces K which bear these measures, and how does one describe closeness between such K's?

I hope that our parade of familiar objects viewed in terms of their associated measures dm will have persuaded you that the "length of a potato" is a useful notion, and that these lower dimensional terms in the measure of a solid are simple enough to be taught in elementary calculus. I have performed the experiment; some of my students enjoyed it.

I have left one mystery to the end: what is the actual value of the length of a ball? You can work it out by calculating the volume of a ball of radius $R + S$ by applying Steiner's formula to a ball of radius S. Or you can use our formulas for solids of revolution. Or you can use the integral-geometric approach. It's twice the diameter.

BIBLIOGRAPHY

There is a vast literature on this topic, under the headings of differential geometry, Riemannian geometry, convexity, geometric measure, integral geometry, and more. As an outsider to all of these, I found three sources most helpful. One is conversation with Bill Lawvere; indeed my own investigations began when he and I asked ourselves: in what sense is the boundary of a solid the derivative of a solid? (I should add that he does not profess to be an expert in the areas listed, any more than I do.) The other two most helpful sources are listed below; Federer's paper gives complete proofs of most of what we have asserted, for the rather general case of "sets of positive reach in Euclidean space", while Hadwiger's book is a leisurely elementary account of the basic notions, especially for polyhedra.

Federer, H. Curvature Measures, Trans. Amer. Math. Soc. 93 (1959) 418 - 491.

Hadwiger, H. Vorlesungen über Inhalt, Oberfläche, und Isoperimetrie, Springer-Verlag, Heidelberg, 1957.