CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C. I. M. E.)

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AXIOMATIC SHEAF THEORY : SOME CONSTRUCTIONS AND APPLICATIONS

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0. Introduction

After having thought for some time about what it was that I wanted to say in these lectures, I finally decided they should serve two purposes, and two purposes only. First, I wanted to provide a leisurely, and reasonably complete, introduction to the axiomatic theory of sheaves developed recently by F. W. Lawvere and myself. Grothendieck, and those around him, have long maintained (see [11]) that in sheaf theory it is the topos itself - i.e., the whole category of sheaves - that is important, and not the site, or small category, from which it is derived. He himself, however, had never consequently developed this point of view. Thus, this was our first goal, and it is the one I would like to concentrate on here. Later, we began to think of the notion of topos as a kind of set theory useful for dealing with many kinds of "sets" other than just "abstact" sets. Though this is perhaps the most interesting aspect of topos, I shall hardly mention it here; I hope the interested reader will consult [4] or [9] for more information. After developing the basic general theory, we will turn to the more specialized topics whose exposition was my second aim. Here, among several possible ways of constructing topos, I would like to describe thesis from the topos point of view. Thus, this section, as well as the preceding, may be considered background material for [10]. The Continuum Hypothesis forms the subject of the last section, so these lectures are independent of [10], though the reader might want to consult this for more details.

Finally, I would like to thank the organizers of C.I.M.E., in particular Professor Paolo Salmon, for inviting me to give these lectures, and the participants of the conference for the interest and fortitude they showed in attending them.

1. Examples of topos

Before giving the formal definition of a topos, I would like to consider three basic examples that we will keep coming back to in the course of the lectures. The first is the category of sets. Let us denote this by $\underline{S}$. Later we will make precise what we mean by a model of the category of sets, but for now think of $\underline{S}$ as, say, the category of naive sets and functions. The notation here is standard, we write 0 for the empty set (the initial object), 1 for the one point set (the terminal object), etc. More generally, let $\underline{C}$ be a small category -

i.e., $\underline{C}$ is a category in $\underline{S}$, which we write as $\underline{C} \in \mathrm{Cat}(\underline{S})$. Then we can form $\underline{S}^{\underline{C}^{op}}$, the category of contravariant set-valued functors on $\underline{C}$, and this is our second example. If $C \in \underline{C}$, let us write simply C for the representable functor $\underline{C}(\ ,C)$. Then the Yoneda lemma may be written as:

$$\frac{C \longrightarrow F}{1 \longrightarrow F(C)}$$

This should be read: "The natural transformations from the representable functor C to the functor F are in natural $1-1$ correspondence with the elements of $F(\underline{C})$". I recall that in $\underline{S}^{\underline{C}^{op}}$ limits and colimits are computed pointwise - i.e., object by object in $\underline{S}$ - and that every functor is a (canonical) colimit of representables. For proofs of these, as well as other basic facts about functor categories, it is probably good to consult [11], since this is also a reference for some of the later material. We will be especially interested in the case $\underline{C} = \underline{P}$, a category arising from a partially ordered set P in $\underline{S}$ (put $p \longrightarrow q$ iff $p \leq q$). As an example of this, let T be a topological space, and write $\underline{T}$ for the category of open subsets of T. Thus, category of set valued presheaves on T. It has a reflective sub-category of particular interest, which forms our third example, called the category of sheaves on T. I recall that a sheaf is a functor F such that if $\{U_i\}_{i\in I}$ is a cover of U, then

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \rightrightarrows \prod_{(i,j)\in I\times I} F(U_i \quad U_j)$$

is an equalizer (all maps are induced by restriction, which is what we call the action of morphisms on F in the case of presheaves). Thus, a sheaf is a presheaf for which compatible local existence implies unique global existence. In

$$\text{Sheaves}(T) \underset{i}{\overset{a}{\leftrightarrows}} \underline{S}^{\underline{T}^{op}}$$

we call the reflection a the associated sheaf functor, and we write i for the inclusion. For the definition of a, as well as for basic properties of sheaves see [3]. Here I only want to note that if U is an open set of T then the representable functor U is a sheaf. Also, recall that the category Sheaves(T) is equivalent to the category of éspaces étalés over T - these are local homeomorphisms $X \longrightarrow T$ with continuous triangles as morphisms. The equivalence is given in detail in [3].

Let us consider now several properties that these categories have in common. For one thing, they all have finite limits and colimits. I recall that this is equivalent to the existence of a terminal object 1, an initial object 0, a pullback for every pair of maps with common codomain, and a pushout for every pair of maps with common domain. We write pullbacks as

$$\begin{array}{ccc} f^*X & \xrightarrow{\pi_2} & X \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow{f} & Y \end{array}$$

or

$$\begin{array}{ccc} Z \times_Y X & \xrightarrow{\pi_2} & X \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow{f} & Y \end{array}$$

Of course, each of our examples really has more limits and colimits than this - there is a limit or colimit for every diagram indexed by a set - but this is a second order property and is due to the fact that each is defined over $\underline{S}$ (in a sense to be made precise later). Since we want to consider only <u>intrinsic</u> properties, for reasons that will become clear later, we will simply ignore these extra limits and colimits.

For the second property, let us consider $\underline{S}$ first. Here, for each pair X, Y $\underline{S}$ we have a set X^Y - the set of functions from Y to X - which has the universal property that for any Z,

$$\frac{Z \longrightarrow X^Y}{Z \times Y \longrightarrow X}$$

(Again, read this as: "The maps from Z to X^Y are in natural 1 - 1 correspondence with the maps from $Z \times Y$ to X".) More precisely, there are natural transformations $ev: X^Y \times Y \longrightarrow X$ and $\eta : Z \longrightarrow (Z \times Y)^Y$ such that the correspondence

$$Z \xrightarrow{f} X^Y \rightsquigarrow Z \times Y \xrightarrow{f \times Y} X^Y \times Y \xrightarrow{ev} X$$

and

$$Z \times Y \xrightarrow{g} X \rightsquigarrow Z \xrightarrow{\eta} (Z \times Y)^Y \xrightarrow{g^Y} X^Y$$

is 1 - 1. These are obvious for $\underline{S}$. Namely,

$$\eta(z)(y) = (z,y)$$
$$ev(f,y) = f(y)$$

The existence of such an exponential object is what we mean by saying $\underline{S}$ is <u>cartesian closed</u>. Note that cartesian closedness implies that for any diagram $\{Z_i\}$

in $\underline{S}$, the canonical map

$$\underset{i}{\underrightarrow{\lim}}(Z_i \times Y) \longrightarrow (\underset{i}{\underrightarrow{\lim}} Z_i) \times Y$$

is an isomorphism - since $(\ ) \times Y$ has the right adjoint $(\ )^Y$.

Now consider $\underline{S}^{\underline{C}^{op}}$. If X and Y are functors, what could X^Y be? Well, if X^Y exists, then, in particular, for a representable functor C we must have

$$\frac{C \longrightarrow X^Y}{C \times Y \longrightarrow X}$$

But the natural transformations from C to X^Y are in natural 1 - 1 correspondence with the elements of $X^Y(C)$, so that if X^Y exists, its value at $C \in \underline{C}$ is forced. Let us then try the definition

$$X^Y(C) = \underline{S}^{\underline{C}^{op}}(C \times Y, X),$$

with morphisms acting in the obvious way. To show that this definition works, we use the fact that for any diagram $\{Z_i\}$ in $\underline{S}^{\underline{C}^{op}}$, the canonical map

$$\underrightarrow{\lim}(Z_i \times Y) \longrightarrow (\underrightarrow{\lim} Z_i) \times Y$$

is an isomorphism. This is true in $\underline{S}^{\underline{C}^{op}}$ because it is true in $\underline{S}$. Now for an arbitrary $Z \in \underline{S}^{\underline{C}^{op}}$ write Z as a colimit of representables; $Z = \varinjlim C_i$. Then

$$\begin{aligned}
\underline{S}^{\underline{C}^{op}}(Z, X^Y) &= \underline{S}^{\underline{C}^{op}}(\varinjlim C_i, X^Y) \\
&\simeq \varprojlim \underline{S}^{\underline{C}^{op}}(C_i, X^Y) \\
&\simeq \varprojlim \underline{S}^{\underline{C}^{op}}(C_i \times Y, X) \\
&\simeq \underline{S}^{\underline{C}^{op}}(\varinjlim (C_i \times Y), X) \\
&\simeq \underline{S}^{\underline{C}^{op}}((\varinjlim C_i) \times Y, X) \\
&= \underline{S}^{\underline{C}^{op}}(Z \times Y, X)
\end{aligned}$$

Hence $\underline{S}^{\underline{C}^{op}}$ is also cartesian closed.

How about Sheaves(T)? Well, we have just seen that the category of presheaves $\underline{S}^{\underline{T}^{op}}$ is cartesian closed, and it is easy to verify that if F is a sheaf and G is <u>any</u> presheaf, then F^G is a sheaf. This holds in particular when G is a sheaf, so Sheaves(T) is also cartesian closed.

Let us return to $\underline{S}$ and consider the two point set $2 = 1 + 1$. This is a simple set, but has a rather remarkable universal property. Namely, let us label its elements "true" and "false", and consider $1 \xrightarrow{\text{true}} 2$. Suppose $X' \rightarrowtail X$ is an arbitrary monomorphism (we will always denote monomorphism by $\rightarrowtail$ and epimorphism by $\twoheadrightarrow$).

If we define $\varphi : X \longrightarrow 2$ by $\varphi(x) = \text{true}$ iff $x \in X'$, then φ is the unique map such that

$$\begin{array}{ccc} X' & \longrightarrow & 1 \\ \downarrow & & \downarrow \text{true} \\ X & \xrightarrow[\varphi]{} & 2 \end{array}$$

is a pullback. φ is usually called the characteristic map of X'. Thus, in $\underline{S}$ there is a universal monomorphism that classifies all others by pullback along a unique characteristic map.

Does this also hold in $\underline{S}^{\underline{C}^{op}}$? That is, is there a functor Ω together with a natural transformation $1 \xrightarrow{\text{true}} \Omega$ having the above universal property? As with exponentiation if such an Ω exists, then, in particular, if C is representable we must have

$$\frac{C \longrightarrow \Omega}{R \rightarrowtail C}$$

and this again forces the definition of $\Omega(C)$:

$$\Omega(C) = \left\{ R \rightarrowtail C \text{ in } \underline{S}^{\underline{C}^{op}} \right\}$$

(Such subobjects $R \rightarrowtail C$ are called "cribles", or sieves in English.) If $\alpha ; C \longrightarrow C'$ is a map in $\underline{C}$, let

$$\Omega(\alpha) : \Omega(C') \longrightarrow \Omega(C)$$

be the map that sends $R' \rightarrowtail C'$ into $\alpha^*(R') \rightarrowtail C$. Define $1 \xrightarrow{\text{true}} \Omega$ by true $(C) = C \xrightarrow{\text{id}} C$ - the identity subfunctor of C. Now suppose $X' \rightarrowtail X$ is an arbitrary monomorphism, and suppose $\varphi : X \longrightarrow \Omega$ is such that

$$\begin{array}{ccc} X' & \longrightarrow & 1 \\ \downarrow & & \downarrow \text{true} \\ X & \xrightarrow[\varphi]{} & \Omega \end{array}$$

is a pullback. Then if $C \in \underline{C}$, and $x \in X(C)$, $\varphi(C)(x)$ is some $R \rightarrowtail C$, and $\alpha : C' \longrightarrow C$ is in $R(C')$ iff $\alpha^*(R) = \text{true}(C')$, i.e. iff $\Omega(\alpha)(R) = \text{true}(C')$. But

$$\begin{array}{ccc} X(C) & \xrightarrow{\varphi(C)} & \Omega(C) \\ X(\alpha)\downarrow & & \downarrow \Omega(\alpha) \\ X(C') & \xrightarrow{\varphi(C')} & \Omega(C') \end{array}$$

commutes, so the above holds iff $X(\alpha)(x) \in X'(C')$. If we represent x as a natural transformation $x : C \longrightarrow X$, then $X(\alpha)(x) \in X'(C')$ iff the diagram

$$\begin{array}{ccccc} C' & \longrightarrow & C \times_X X' & \longrightarrow & X' \\ \text{id}\downarrow & & \downarrow & & \downarrow \\ C' & \xrightarrow{\alpha} & C & \xrightarrow{x} & X \end{array}$$

is a pullback, iff $\alpha \in C \times_X X'(C')$. Thus we must have

$$\varphi(C)(x) = C \times_X X' \rightarrowtail C \qquad .$$

On the other hand, if we define φ this way then clearly φ is natural, and clearly

$$\begin{array}{ccc} X' & \longrightarrow & 1 \\ \downarrow & & \downarrow \text{true} \\ X & \xrightarrow[\varphi]{} & \Omega \end{array}$$

is a pullback. Thus $1 \xrightarrow{\text{true}} \Omega$ classifies subobjects in $\underline{S}^{C^{op}}$.

Now consider Sheaves(T). Here again, if such an Ω exists, we must have

$$\frac{U \longrightarrow \Omega}{S \rightarrowtail U}$$

for an open set $U \subseteq T$. However, thinking in terms of éspaces étalés it is clear that a subobject $S \rightarrowtail U$ is nothing but an open set $V \subseteq U$. Thus we are forced to define

$$\Omega(U) = \{ \text{open } V \subseteq U \} \qquad ,$$

letting morphisms act by intersection. Again, true:

$1 \longrightarrow \Omega$ picks out for each U, the whole open subset U. This Ω is a sheaf, for it is the sheaf of sections of $T \times \mathcal{O} \xrightarrow{\pi_1} T$, where $\mathcal{O}$ is the Sirpinski space consisting of one open and one closed point. An argument simelar to the one for $\underline{S}^{\underline{C}^{op}}$ shows that it has the correct universal property.

Thus we have three categorical properties shared by these three examples. There are, of course, others For example, each has an object of generators indexed by an object of $\underline{S}$. Such properties are, again, second order, however, and are tied to the fixed base $\underline{S}$, so we ignore them. One other intrinsic property that they do all share is worth mentioning, though. Namely, they all satisfy an axiom of infinity. This we express by asserting the existence of a natural number object ω together with an initial point $1 \xrightarrow{o} \omega$ and a successor function $\omega \xrightarrow{s} \omega$. Furthermore, we require that ω be universal with respect to this data. That is, for any other X provided with $1 \xrightarrow{xo} X$ and $X \xrightarrow{t} X$, there is a unique map $\omega \longrightarrow X$ such that

$$\begin{array}{ccccc} 1 & \xrightarrow{o} & \omega & \xrightarrow{s} & \omega \\ & \searrow^{xo} & \downarrow & & \downarrow \\ & & X & \xrightarrow{t} & X \end{array}$$

commutes. In $\underline{S}$ ω is the set of natural numbers with the element o and the successor function, in $\underline{S}^{\underline{C}^{op}}$ it is the constant functor at the ω of $\underline{S}$, and in Sheaves(T) it is the sheaf associated to the ω of $\underline{S}^{T^{op}}$.

Neglecting this last point for the time being, let us define a topos to be a category $\underline{E}$ that has finite limits and colimits, a subobject classifier $1 \xrightarrow{\text{true}} \Omega$, and is cartesian closed. Now although these properties will be sufficient for most of what we will do, at least at the beginning, we should still recognize that there exist topos in this sense with undesirable properties. For example, the category of finite sets is a topos. But here, though the abelian group objects form an abelian category, they neither have enough injectives (in fact they have no injectives) nor do they have any free objects. The latter problem can be solved by adding ω, though it is not clear if this also solves the former.

2. Properties of one topos

Now that we have the axioms for a topos, let us see what follows from them. First, notice that the presence of Ω implies immediately that $\underline{E}$ is balanced - i.e., any map that is a monomorphism and an epimorphism is an isomorphism. This is so because, first,

$$1 \xrightarrow{\text{true}} \Omega \underset{\text{id}}{\overset{\longrightarrow 1 \xrightarrow{\text{true}}}{\longrightarrow}} \Omega$$

is an equalizer, and hence, since every monomorphism is a pullback of true, every monomorphism is an equalizer. Thus, if $X \xrightarrow{f} Y$ is monic and epic it is an equalizer and epic, hence an isomorphism.

Recall that an <u>equivalence relation</u> an $X \in \underline{E}$ is a subobject $R \rightarrowtail X \times X$ that is reflexive, symmetric, and transitive. That is, for each $Y \in \underline{E}$,

$$\underline{E}(Y, R) \rightarrowtail \underline{E}(Y, X) \times \underline{E}(Y, X)$$

is an equivalence relation, or one can express this directly in terms of pullbacks in $\underline{E}$. In any case, let $\varphi : X \times X \longrightarrow \Omega$ be the characteristic map of $R \rightarrowtail X \times X$, and suppose $\bar{\varphi} : X \longrightarrow \Omega^X$ is its transpose - $\bar{\varphi}$ is the map that assigns to each "element" of X its equivalence class mod R. Then we can prove - and this is somewhat tricky - that if $R \overset{\rho_1}{\underset{\rho_2}{\rightrightarrows}} X$ are the composites $R \rightarrowtail X \times X \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X$, then

$$R \overset{\rho_1}{\underset{\rho_2}{\rightrightarrows}} X \xrightarrow{\bar{\varphi}} \Omega^X \quad .$$

is a pullback diagram. Thus, every equivalence relation is the kernel pair of a map, which is the statement that <u>equivalence relations are effective</u>.

A <u>partial map</u> from X to Y is a diagram

$$\begin{array}{ccc} X' & \longrightarrow & Y \\ \downarrow & & \\ X & & \end{array}$$

($X' \rightarrowtail X$ is called the <u>domain</u> of the map), and an important property of topos is that <u>partial maps are representable</u>. Namely, for each $Y \in \underline{E}$ there is a monomorphism $Y \rightarrowtail \tilde{Y}$ such that given any partial map as above, there exists a unique map $X \longrightarrow \tilde{Y}$ such that

$$\begin{array}{ccc} X' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & \tilde{Y} \end{array}$$

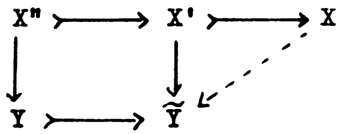
is a pullback. Note that $1 \overset{true}{\rightarrowtail} \Omega$ is $1 \rightarrowtail \tilde{1}$, so this is the strong form of the axiom asserting the existence of Ω. Now, how to define $\tilde{Y}$? Well, first notice that maps $X \longrightarrow \Omega^Y$ classify arbitrary subobjects of $X \times Y$. Thus, an object classifying subobjects of $X \times Y$, whose first projection is monic should be a part of Ω^Y. In fact, denote the characteristic map of $\Delta : Y \rightarrowtail Y \times Y$ by $\textcircled{=}_Y : Y \times Y \longrightarrow \Omega$, and call its transpose "singleton" : $\{\ \} : Y \rightarrowtail \Omega^Y$ ($\{\ \}$ is monic by the previous result). Consider the graph of $\{\ \} : \langle \{\ \}, id \rangle : Y \longrightarrow \Omega^Y \times Y$, and let $\varphi : \Omega^Y \times Y \longrightarrow \Omega$ be its characteristic map. Then

$$\tilde{Y} \rightarrowtail \Omega^Y \underset{id}{\overset{\bar{\varphi}}{\rightrightarrows}} \Omega^Y$$

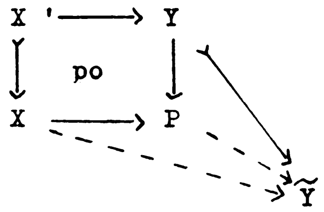
is an equalizer. One checks that $Y \xrightarrow{\{\}} \Omega^Y$ equalizes $\bar{\varphi}$ and id, which provides the sought for $Y \rightarrowtail \tilde{Y}$. Notice that in $\underline{S}$ the above equation picks out those subsets Y' of Y satisfying

$$Y' = \{y \mid Y' = \{y\}\}$$

These are precisely the elements of Y - the singletons - together with the empty set - the element "undefined". Thus, in $\underline{S}$ $\tilde{Y}$ is $Y + 1$. $\tilde{Y}$ is injective - consider



thus, $Y \rightarrowtail \tilde{Y}$ provides a functorial embedding of each Y into an injective - $\tilde{Y}$ is <u>not</u> an injective envelope, however. Also, using $\tilde{Y}$ one quickly shows that the pushout of a monomorphism is a monomorphism, for consider



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which shows that $Y \longrightarrow P$ is monic. Note that such a pushout is also a pullback, because the outer square is.

If the existence of $\tilde{Y}$ for each Y is the strong form of Ω , then the strong form of exponentiation is the following. In any category with pullbacks, a map $f : X \longrightarrow Y$ induces a functor

$$\underline{E}/_Y \xrightarrow{f^*} \underline{E}/_X$$

given by pulling an object $Z \longrightarrow Y$ back along f to an object $f^*Z \longrightarrow X$. This functor always has a left adjoint $\sum_f$ given by composition with f - the statement $\sum_f \vdash f^*$ is just the universal property of the pullback. In a topos, however, <u>f has a right adjoint</u> $\prod_f$. The notation comes from the case $\underline{E} = \underline{S}$. Namely, suppose we have

$$\begin{array}{ccc} E & & \\ {\scriptstyle p}\downarrow & & \\ X & \xrightarrow{f} & Y \end{array}$$

Then what is $(\sum_f E)_y$ - the fibre of $E \xrightarrow[p]{} X \xrightarrow[f]{} Y$ above $y \in Y$? Well, it is the pullback

$$\begin{array}{ccc} \sum_{x \in f^{-1}(y)} E_x & \longrightarrow & E \\ \downarrow & & \downarrow p \\ f^{-1}(y) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ 1 & \xrightarrow{y} & Y \end{array} \quad .$$

Thus $(\sum_f E)_y = \sum_{x \in f^{-1}(y)} E_x$.
Similarly, whatever $\prod_f E \longrightarrow Y$ may be, what must be its elements above $y \in Y$? Well, these elements are maps

$$\begin{array}{ccc} 1 & \longrightarrow & \prod_f E \\ & {}_y\searrow \quad \swarrow & \\ & Y & \end{array} \quad ,$$

which are in 1 - 1 correspondence with maps

$$\begin{array}{ccc} f^{-1}(y) & \longrightarrow & E \\ & \searrow \quad \swarrow {}_p & \\ & X & \end{array} \quad .$$

But these latter constitute the elements of $\prod_{x \in f^{-1}(y)} E_x$.
Thus

$$(\prod_f E)_y = \prod_{x \in f^{-1}(y)} E_x \quad ,$$

and one can use this as a definition in $\underline{S}$. In an arbitrary topos $\underline{E}$. Consider

$$\begin{array}{ccc} E & & \\ {\scriptstyle p}\downarrow & & \\ X & \xrightarrow{f} & Y \end{array} .$$

The partial map

$$\begin{array}{ccc} X & \xrightarrow{id} & X \\ {\scriptstyle \langle f,\, id\rangle}\downarrow & & \\ Y \times X & & \end{array}$$

gives a map $Y \times X \longrightarrow \tilde{X}$ with transpose $Y \longrightarrow \tilde{X}^{X}$. Then

$$\begin{array}{ccc} \prod_f E & \longrightarrow & \tilde{E}^{X} \\ \downarrow & & \downarrow {\scriptstyle \tilde{p}^{X}} \\ Y & \longrightarrow & \tilde{X}^{X} \end{array}$$

is a pullback. $\prod$ is closely related to exponentiation. Namely, for $X \in \underline{E}$, let us write $\prod_X$ and $\sum_X$ for $\prod$ and $\sum$ along the canonical map $X \longrightarrow 1$. Now consider the two adjoint pairs

$$\underline{E} \underset{\sum_X}{\overset{(\ )\times X}{\rightleftarrows}} \underline{E}/X \underset{(\ )\times X}{\overset{\prod_X}{\rightleftarrows}} \underline{E} .$$

The bottom composite is just ( ) x X (without its structural map π_2), hence the top is $(\ )^X$. Thus, the existence of $\prod_X$ for each X implies $\underline{E}$ is cartesian closed (finite limits is a blanket assumption). On the other hand, if $E \xrightarrow{p} X$, then

$$\begin{array}{ccccc} \prod_X E & \rightarrowtail & E^X & \xrightarrow{p^X} & X^X \\ & & & \searrow \quad \nearrow_{id} & \\ & & & 1 & \end{array}$$

is an equalizer, so we can recover $\prod_X$ from exponentiation. Thus $\prod_X$ and $(\ )^X$ are equivalent in the presence of finite limits. In the same way, $\prod_f$ for each $f : X \longrightarrow Y$ is equivalent to the assumption that each $\underline{E}/_Y$ is cartesian closed. I remark, however, that this does <u>not</u> follow from finite limits and exponentiation in $\underline{E}$, as is shown by the example of <u>Cat</u>. Another point is that if we have $\prod$ and Ω, then we have an easy definition of $\tilde{X}$. Namely, in the diagram

$$\begin{array}{ccc} X & & \prod_{true} X \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{true} & \Omega \end{array}$$

$\prod_{true} X$ is $\tilde{X}$, and $\tilde{X} \longrightarrow \Omega$ is the map that assigns to a partial map its domain.

Since $\prod_f$ exists, it follows that f^* preserves colimits for any map f. This is the statement that <u>colimits</u> <u>are</u> <u>universal</u>. In particular, f^* preserves coequalizers, which, together with finite limits and effective equivalence relations, enables one to prove that $Ab(\underline{E})$, the category of abelian group objects in $\underline{E}$, is abelian. In addition, this has the agreeable consequence that given any $X \xrightarrow{f} Y$, then in the factorization

$$\begin{array}{ccccc} X \times_Y X & \overset{p_2}{\underset{p_1}{\rightrightarrows}} & X & \xrightarrow{f} & Y \\ & & & \searrow\!\!\!\searrow & \uparrow{\scriptstyle i} \\ & & & & Q \end{array}$$

i is monic. Thus every epimorphism is a coequalizer, and we have a unique coequalizer, equalizer factorization for any map. Also, $0 \longrightarrow Y$ being the empty colimit over Y, it follows that for each $f : X \longrightarrow Y$,

$$\begin{array}{ccc} 0 & = & 0 \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback. But then, if $X \longrightarrow 0$, it follows that

$$\begin{array}{ccc} 0 & = & 0 \\ \downarrow & & \| \\ X & \longrightarrow & 0 \end{array}$$

is a pullback, so $0 \xrightarrow[\simeq]{} X$, which is the statement 0 is strict. (This also follows easily from cartesian closedness alone). If 0 is strict, though, then each $0 \longrightarrow X$ is monic, so in the pushout

$$\begin{array}{ccc} 0 & \rightarrowtail & Y \\ \downarrow & & \downarrow \\ X & \rightarrowtail & X + Y \end{array}$$

the injections of the coproduct are monic, and the diagram is also a pullback - i.e., coproducts are disjoint.

Relative to a fixed $\underline{S}$, a theorem of Giraud shows that a Grothendieck topos can be characterized as a category $\underline{E}$ having:

Finite limits and $\underline{S}$ - indexed disjoint, universal coproducts; universally effective equivalence relations - this means that equivalence relations have coequalizers whose kernels they are, and f^* preserves coequalizers of equivalence relations for any f;

an object of generators indexed by an object of $\underline{S}$.

Thus, any of these properties that does not involve $\underline{S}$ holds for our intrinsic topos. We will come back to this point, and to Grothendieck topos, shortly.

Let us turn our attention, for the moment, away from the whole topos $\underline{E}$, and concentrate on properties of the object Ω. Among other reasons, Ω is remarkable in that to define a map $\varphi : X \longrightarrow \Omega$, one need only know where φ takes the single value "true" - all the many other possible truth values take care of themselves. A good illustration of this is the derivation of the algebraic structure present on Ω. First we are given the "element" true : $1 \longrightarrow \Omega$. The characteristic map of $0 \rightarrowtail 1$ gives the "element" false : $1 \longrightarrow \Omega$. We can define an operation "conjunction", written $\wedge : \Omega \times \Omega \longrightarrow \Omega$, by specifying where it should take the value "true", which, as everyone knows, is at the pair $\langle \text{true}, \text{true} \rangle$: $1 \rightarrowtail \Omega \times \Omega$. Similarly, "disjunction", written $\vee : \Omega \times \Omega \longrightarrow \Omega$ is defined by letting it be "true" precisely at $(1 \times \Omega) \cup (\Omega \times 1) \rightarrowtail \Omega \times \Omega$. The equalizer

$$\Omega_0 \rightarrowtail \Omega \times \Omega \overset{\wedge}{\underset{\pi_1}{\rightrightarrows}} \Omega$$

defines the order relation $\Omega_0 \rightarrowtail \Omega \times \Omega$ on Ω, which, in turn, defines, as above, the operation "implication", written $\Rightarrow : \Omega \times \Omega \longrightarrow \Omega$. With this structure Ω becomes a Heyting algebra object in $\underline{E}$. For category theorists this means that with the $\underline{E}$ - category structure induced by the partial ordering $\Omega_0 \rightarrowtail \Omega \times \Omega$, Ω is finitely complete (with product $\wedge$ and terminal object "true"), finitely cocomplete (with coproduct $\vee$ and initial object "false"), and cartesian closed ($\Rightarrow$ is the exponential for $\wedge$). This structure on Ω induces a Heyting algebra structure on the lattice of subobjects of any object, and the cartesian closed property can be expressed as

$$\frac{\alpha \wedge \varphi \longrightarrow \psi}{\alpha \longrightarrow (\varphi \Rightarrow \psi)}$$

We can further define the operator "negation" - $\neg : \Omega \longrightarrow \Omega$ - as being the characteristic map of $1 \xrightarrow{\text{false}} \Omega$. The propositional calculus thus arising is usually intuitionistic in the sense that $\neg\neg \neq \text{id}$. For example, if T is Hausdorff, then in Sheaves(T) $\neg\neg = \text{id}$ iff T is discrete. If $\neg\neg = \text{id}$ we call $\underline{E}$ Boolean. Equivalent conditions are: $\langle \text{true}, \text{false} \rangle$:

$1 + 1 \longrightarrow \Omega$ is an isomorphism, Ω is a Boolean algebra object, or given any $X' \rightarrowtail X$, there exists an $X'' \rightarrowtail X$ such that the map $X' + X'' \longrightarrow X$ is an isomorphism. However, even if $\neg\neg \neq \text{id}$, it has important properties and we will return to it shortly.

In addition to this propositional structure on Ω we can also quantify along any map $f : X \longrightarrow Y$. This means that $\Omega^f : \Omega^Y \longrightarrow \Omega^X$ has both a left adjoint $\exists f$ and a right adjoint $\forall f$. Thus, actual subobjects exist for any formula of higher order logic with bounded quantifiers. In addition, for each X there are operations

$$\Omega^{\Omega^X} \xrightarrow{\text{sup}} \Omega^X \quad \text{and} \quad \Omega^{\Omega^X} \xrightarrow{\text{inf}} \Omega^X,$$

making each Ω^X $\underline{E}$-complete and cocomplete. That is, for any family of subobjects of X that is indexed by an object of $\underline{E}$ there exists both a sup and an inf. We will not stress this internal logic here, but merely mention it in passing. For more details see [4] or [9].

3. Topologies

We have examined some consequences of the axioms within one topos, let us now see how to construct new topos from old. Perhaps the most important way to obtain new topos is by passing to sheaves with respect to a topology, so we will concern ourselves with topologies and sheaves for the next two lectures.

Let us begin by considering a topological space T, and asking what are sheaves on T, and how do we get them from the presheaves $\underline{S}^{\underline{T}^{op}}$? We will first examine Ω, and compute some of the operations we derived last time. Recall that in $\underline{S}^{\underline{T}^{op}}$, if $U \subseteq T$ is open then

$$\Omega(U) = \{ R \rightarrowtail U \} .$$

What is such a crible $R \rightarrowtail U$? If V is open, then $R(V)$ is either 0 or 1, and if $R(V) = 1$, then $V \subseteq U$, and if $V' \subseteq V$ we must have $R(V') = 1$, since $R(V) \longrightarrow R(V')$. Thus we can identify R with the collection of $V \subseteq U$ such that $R(V) = 1$. R thus becomes a filter of open subsets $V \subseteq U$ such that if $V \in R$ and $V' \subseteq V$, then $V' \in R$. Note that any family $\{U_i\}_{i \in I}$ generates a crible $R[U_i]$ such that $V \in R[U_i]$ iff for some $i \in I$, $V \subseteq U_i$. These considerations, by the way, as well as those that follow, apply to any partially ordered set, and not just to $\underline{T}$.

As we have seen, $1 \xrightarrow{\text{true}} \Omega$ picks out, for each U, the identity crible, and it is easy to check that

$1 \xrightarrow{\text{false}} \Omega$ picks out the empty crible $0 \rightarrowtail U$ for each U (note that this is <u>not</u> the crible represented by the empty set). What about $\wedge : \Omega \times \Omega \longrightarrow \Omega$? Well, if $R \rightarrowtail U$ and $R' \rightarrowtail U$, then

$$\begin{array}{ccccc} R \cap R' & \longrightarrow & 1 & = & 1 \\ \downarrow & & \downarrow \langle \text{true}, \text{true} \rangle & & \downarrow \text{true} \\ U & \xrightarrow{\langle R, R' \rangle} & \Omega \times \Omega & \longrightarrow & \Omega \end{array}$$

is a pullback, so

$$R \wedge R' = R \cap R' = \{ W \mid W \in R \text{ and } W \in R' \}.$$

Similarly, for $\vee : \Omega \times \Omega \longrightarrow \Omega$ we have

$$R \vee R' = R \cup R' = \{ W \mid W \in R \text{ or } W \in R' \}.$$

For $\Rightarrow : \Omega \times \Omega \longrightarrow \Omega$, the diagram

$$\begin{array}{ccccc} R \Rightarrow R' & \longrightarrow & \Omega_o & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow \text{true} \\ U & \xrightarrow{\langle R, R' \rangle} & \Omega \times \Omega & \xrightarrow{\Rightarrow} & \Omega \end{array}$$

is a pullback. Thus, at V $(R \Rightarrow R')(V) = 1$ iff

$R|V < R'|V$. Therefore,

$$R \Rightarrow R' = \{V \mid \forall W \subseteq V, \ \ W \in R \text{ implies } W \in R'\} .$$

Let us return to the question, what are sheaves? Well, they are presheaves such that compatible local existence of a section implies unique global existence. Compatible local existence means on a cover, so sheaves are presheaves with a special property on covers. Thus, we must know when a collection $U_i \subseteq U$ covers U. How can we measure this? Well, we can consider the crible generated by the union of the U_i. Then the U_i cover iff this is the identity crible U. More formally, let us define $t : \Omega \longrightarrow \Omega$ as follows. If U is open, let

$$t_U : \Omega(U) \longrightarrow \Omega(U)$$

be given by $t_U(R) = [\bigcup R]$ = the crible generated by the union of R. To check naturality, suppose $V \subseteq U$. We want

$$\begin{array}{ccc} \Omega(U) & \xrightarrow{t_U} & \Omega(U) \\ \downarrow & & \downarrow \\ \Omega(V) & \xrightarrow{t_V} & \Omega(V) \end{array}$$

to commute, i.e., we want $\left[\bigcup R\right] \cap V = \left[\bigcup(R \cap V)\right]$ for each $R \rightarrowtail U$. But $W \in \left[\bigcup R\right] \cap V$ iff $W \subseteq V$ and $W \subseteq \bigcup R$ iff $W \subseteq (\bigcup R) \cap V = \bigcup(R \cap V)$ iff $W \in \left[\bigcup (R \cap V)\right]$. Then, we have a natural transformation

$$t : \Omega \longrightarrow \Omega \qquad .$$

which determines, and is determined by, a pullback

$$\begin{array}{ccc} J & \longrightarrow & 1 \\ \downarrow & & \downarrow \text{true} \\ \Omega & \xrightarrow{\ t\ } & \Omega \end{array} \qquad .$$

Here $J(U) = \left\{ R \rightarrowtail U \,\middle|\, t_U(R) = U \right\}$, but $t_U(R) = \left[\bigcup R\right] = U$ iff $\bigcup R = U$. Thus, J at U consists of the covers of U. Note that we could have defined t as the characteristic map of $J \rightarrowtail \Omega$.

What are the algebraic properties of t? First of all, note that $t_U(R) = R$ iff R is a principal filter, i.e., R is generated by a single open set. Thus, in the equalizer

$$\Omega_t \rightarrowtail \Omega \underset{id}{\overset{t}{\rightrightarrows}} \Omega \qquad ,$$

Ω_t is the Ω for the category of sheaves, as we have seen in the first lecture. This is a general principle, as will become evident later. In any case, it follows that $t \circ \text{true} = \text{true}$, and $t \circ t = t$. Also, we claim

$$\begin{array}{ccc} \Omega \times \Omega & \xrightarrow{t \times t} & \Omega \times \Omega \\ \wedge \downarrow & & \downarrow \wedge \\ \Omega & \xrightarrow{t} & \Omega \end{array}$$

commutes. Indeed, we have, for $R \rightarrowtail U$ and $R' \rightarrowtail U$,

$$\begin{aligned} t_U(R) \wedge t_U(R') &= \{ W \mid W \subseteq \bigcup R \text{ and } W \subseteq \bigcup R' \} \\ &= \{ W \mid W \subseteq (\bigcup R) \cap (\bigcup R') \} \end{aligned}$$

and

$$t_U(R \wedge R') = \{ W \mid W \subseteq \bigcup (R \cap R') \} .$$

But $(\bigcup R) \cap (\bigcup R') = \bigcup (R \cap R')$.

We have now expressed the notion of covering by an endomorphism of Ω with certain algebraic properties. What if one needed, as one does, a more general notion of "topology" than this? Well, Grothendieck's approach - a very natural one - is to axiomatize the notion of covering, making it applicable to an arbitrary small category $\underline{C}$ rather than just to categories of the form $\underline{T}$. Thus, supposing for the moment that $\underline{C}$ has pullbacks, a <u>topology on</u> $\underline{C}$ is given by specifying, for each $C \in \underline{C}$, the "covers" of C. These are certain collections

$\{C_i \longrightarrow C\}$, which should satisfy:

(0.) $C \xrightarrow{id} C$ covers C

(1.) If a family of morphisms $C_i \longrightarrow C$ covers C, and if $\alpha : C' \longrightarrow C$, then the family $C' \underset{C}{\times} C_i \longrightarrow C'$ covers C'.

(2.) If $C_i \longrightarrow C$ covers C, and $C_{ij} \longrightarrow C_i$ covers C_i for each i, then the $C_{ij} \longrightarrow C_i \longrightarrow C$ cover C.

Now, going over to $\underline{S}^{\underline{C}^{op}}$, consideration of the cribles $R \rightarrowtail C$ generated by the covers of C yields the following notion of "topology" (which differs from the preceeding only in that pullbacks are no longer necessary in $\underline{C}$). A <u>topology on $\underline{C}$</u> consists of a choice, for each C, of certain cribles $R \rightarrowtail C$ called "covering" (or, later, "dense"). These should satisfy:

(0.) $C \xrightarrow{id} C$ covers C

(1.) If $R \rightarrowtail C$ covers C, and $\alpha : C' \longrightarrow C$, then $C' \underset{C}{\times} R \longrightarrow C'$ covers C'.

(2.) If $R' \rightarrowtail C$, and $R \rightarrowtail C$ is a cover such that for each $\alpha : C' \longrightarrow C$ in R we have $\alpha^*(R') \rightarrowtail C'$ covers C', then R' covers C.

Given such a topology, let us define a functor J by

$$J(C) = \{\text{covers } R \rightarrowtail C\} \qquad ,$$

where morphisms act by pullback (this makes sense by (1.)). Thus J is a functor, and we have a natural inclusion $J \rightarrowtail \Omega$. Call the characteristic map of $J \rightarrowtail \Omega$, $j : \Omega \longrightarrow \Omega$. One checks now that j has the same algebraic properties as our previous t, namely $j \cdot \text{true} = \text{true}$, $j \cdot j = j$, and $j\varphi \wedge j\psi = j(\varphi \wedge \psi)$. Furthermore, any such j determines a topology J by pulling back true. Thus, the correspondence between Grothendieck topologies on $\underline{C}$ and such endomorphisms of Ω is bijective in the case of $\underline{S}^{\underline{C}^{op}}$. Our approach, then, is to throw away "sites", i.e., small categories, altogether, and concentrate on the j's. After all, the algebraic definition makes sense in <u>any</u> topos.

Thus, if $\underline{E}$ is a topos, a <u>topology in</u> (not on here) $\underline{E}$ is an endomorphism $j : \Omega \longrightarrow \Omega$ satisfying the above equations. There are several topologies present in any topos. Two trivial ones : the <u>discrete topology</u> $id : \Omega \longrightarrow \Omega$, the <u>indiscrete topology</u> $\Omega \longrightarrow 1 \xrightarrow{\text{true}} \Omega$, and the highly non-trivial $\neg\neg : \Omega \longrightarrow \Omega$. The fact that $\neg\neg$ is a topology is a standard fact about Heyting algebras - see, for example, [7] where Heyting algebras are called pseudo-Boolean algebras.

4. Sheaves

Now that we have the notion of a topology, let us discuss sheaves. First we return to $\underline{S}^{\underline{T}^{op}}$. Recall that a presheaf F is separated iff for every cover $\{U_i\}$ of U,

$$F(U) \rightarrowtail \prod_i F(U_i)$$

and a sheaf iff

$$F(U) \rightarrowtail \prod_i F(U_i) \rightrightarrows \prod_{(i,j)} F(U_i \cap U_j)$$

is an equalizer. Going over to covering cribles we claim F is separated iff for every covering $R \rightarrowtail U$,

$$F(U) \rightarrowtail \varprojlim_{V \in R} F(V)$$

and a sheaf iff

$$F(U) \xrightarrow{\simeq} \varprojlim_{V \in R} F(V) \qquad .$$

This is obvious. But $F(U) \simeq \underline{S}^{\underline{T}^{op}}(U, F)$, and since $R \simeq \varinjlim_{V \in R} V$ (as a functor), we have

$$\varprojlim_{V \in R} F(V) \simeq \varprojlim_{V \in R} \underline{S}^{\underline{T}^{op}}(V, F)$$

$$\simeq \underline{S}^{\underline{T}^{op}}(\varinjlim_{V \in R} V, F)$$

$$\simeq \underline{S}^{\underline{T}^{op}}(R, F) \quad .$$

Moreover, the induced morphism

$$F(U) \longrightarrow \varprojlim_{V \in R} F(V)$$

is the map

$$\underline{S}^{\underline{T}^{op}}(U, F) \longrightarrow \underline{S}^{\underline{T}^{op}}(R, F)$$

induced by compostion with $R \rightarrowtail U$. Thus, F is separated iff the above is monic for all covering $R \rightarrowtail U$, and a sheaf iff it is an isomorphism. We can extend the notion of covering to arbitrary monomorphisms $G' \rightarrowtail G$, by saying $G' \rightarrowtail G$ <u>covers</u> iff for all $U \longrightarrow G$, in

$$\begin{array}{ccc} U \times_G G' & \longrightarrow & G' \\ \downarrow & & \downarrow \\ U & \longrightarrow & G \end{array}$$

$U \times_G G' \longrightarrow U$ covers. This is the definition of Grothendieck in the case $\underline{S}^{\underline{C}^{op}}$. What does this mean in terms of $t : \Omega \longrightarrow \Omega$? Well, if $\varphi : G \longrightarrow \Omega$ is the characteristic map of $G' \rightarrowtail G$, then we claim $G' \rightarrowtail G$ covers iff in

$$\begin{array}{ccccc} & & J & \longrightarrow & 1 \\ & \nearrow & \downarrow & & \downarrow \text{true} \\ G & \xrightarrow{\varphi} & \Omega & \xrightarrow{t} & \Omega \end{array} ,$$

φ factors thru J - i.e., iff the subobject determined by $t \cdot \varphi$ is $G \xrightarrow{id} G$. This is clear, for $G \simeq \varinjlim_{U \to G} U$. In terms of this extended notion of covering, we claim F is separated iff for every covering $G' \rightarrowtail G$,

$$\underline{S}^{\underline{T}^{op}}(G, F) \longrightarrow \underline{S}^{\underline{T}^{op}}(G', F)$$

is monic, and a sheaf iff it is bijective. Using the above colimit representation of G, this is easy to prove. Now, however, both the codition on covering in terms of t and sheaf in terms of covering make sense in an arbitrary topos - i.e., they do not use the generators U - and these are the definitions we use.

Thus, let $j : \Omega \longrightarrow \Omega$ be a topology in $\underline{E}$. j induces a closure operator on the lattice of subobjects of any $X \in \underline{E}$. Namely, if $X' \rightarrowtail X$ has characteristic map $\varphi : X \longrightarrow \Omega$, let $\overline{X'} \rightarrowtail X$ be the subobject associated to $j \cdot \varphi$. Note that, since j preserves true, we have

$$\begin{array}{ccc} X' & \dashrightarrow & \overline{X'} \\ & \searrow \quad \swarrow & \\ & X & \end{array} .$$

Call $X' \rightarrowtail X$ <u>dense</u> (previously covering) if $\overline{X'} = X$, and <u>closed</u> if $\overline{X'} = X'$. Note that $X' \rightarrowtail \overline{X'}$ is dense. Both dense and closed subobjects have classifying objects. Indeed, in the pullback

$$\begin{array}{ccc} J & \longrightarrow & 1 \\ \downarrow & & \downarrow \text{true} \\ \Omega & \xrightarrow{j} & \Omega \end{array} ,$$

J classifies dense subobjects, and in the equalizer

$$\Omega_j \rightarrowtail \Omega \underset{id}{\overset{j}{\rightrightarrows}} \Omega ,$$

Ω_j classifies closed subobjects.

We say $Y \in \underline{E}$ is <u>separated</u> iff for any dense $X' \overset{i}{\rightarrowtail} X$, and any pair $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, $fi = gi$ yields

$f = g$. Y is a **sheaf** iff for any $X' \xrightarrow{f'} Y$, in

$$\begin{array}{ccc} X' & \overset{i}{\rightarrowtail} & X \\ {\scriptstyle f'}\downarrow & \swarrow{\scriptstyle f} & \\ Y & & \end{array}$$

there exists a unique $f : X \longrightarrow Y$ such that $fi = f'$. Equivalent, more geometric, conditions are: Y is separated iff $Y \overset{\Delta}{\rightarrowtail} Y \times Y$ is closed and a sheaf iff separated and <u>absolutely closed</u>, i.e., closed in any separated object containing it.

Let us write j-Sheaves($\underline{E}$) for the full subcategory of $\underline{E}$ whose objects are the j-sheaves, and denote by i the inclusion

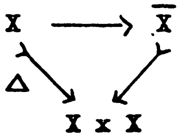
$$j\text{-Sheaves}(\underline{E}) \xrightarrow[i]{} \underline{E} \quad .$$

What we need now is an associated sheaf functor a, i.e., a left adjoint for i. Here, for the first time, our degree of generality begins to show its worth. Namely, we have no infinite limits, no generators, no site. Thus, we are forced to forgo the usual construction of a, and find a simpler, more intrinsic one. This construction is based on the following observations - all extremely simple to prove.

(i.) Ω_j is a sheaf; thus, by a previous remark, it is the Ω for j-Sheaves($\underline{E}$).

(ii.) If Y is a sheaf, then for any X, Y^X is a sheaf.

(iii.) In



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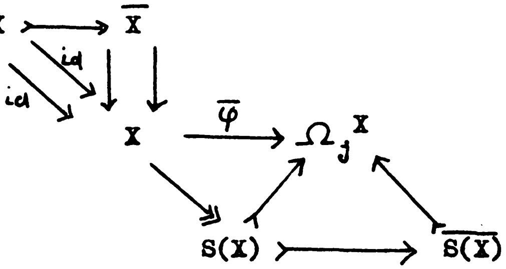
$\overline{X} \rightarrowtail X \times X$ is an equivalence relation.

Now let $\varphi : X \times X \longrightarrow \Omega_j$ be the characteristic map of $\overline{X} \rightarrowtail X \times X$ ($\varphi = j \cdot \textcircled{=}_X$). Then in the diagram



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we have, as before, $\overline{X} \rightrightarrows X$ is the kernel pair of $\overline{\varphi}$ (by (iii.)), and $S(X) \rightarrowtail \Omega_j^X$ is monic, where $S(X)$ is the coequalizer of $\overline{X} \rightrightarrows X$. We claim $S(X)$ is the separated reflection of X. For, if $X \longrightarrow Y$ where Y is separated, then certainly the two composites

$$X \rightarrowtail \overline{X} \rightrightarrows X \longrightarrow Y$$

are equal. Hence, since $X \rightarrowtail \overline{X}$ is dense, it follows that

$$\overline{X} \rightrightarrows X \longrightarrow Y$$

are equal, hence $X \longrightarrow Y$ factors uniquely as

$$\begin{array}{ccc} X & \twoheadrightarrow & S(X) \\ \downarrow & \swarrow & \\ Y & & \end{array} .$$

Also, $S(X)$ is separated since it is a subobject of the separated object Ω_j^X ((i) and (ii.)). Suppose, further, that Y had been a sheaf. Then $S(X) \longrightarrow Y$ factors uniquely across $S(X) \rightarrowtail \overline{S(X)}$, Giving a unique factorization

$$\begin{array}{ccccc} X & \twoheadrightarrow & S(X) & \rightarrowtail & \overline{S(X)} \\ \downarrow & \swarrow & & \swarrow & \\ Y & & & & \end{array} .$$

But $\overline{S(X)}$ is a sheaf, since it is a closed subobject of the sheaf Ω_j^X. Thus, $\overline{S(X)} = a(X)$ - the associated sheaf of X. So we have our adjoint pair

$$\text{j-Sheaves}(\underline{E}) \underset{i}{\overset{a}{\rightleftarrows}} \underline{E} \qquad ,$$

and we have constructed a new topos j-Sheaves($\underline{E}$) from the topology in $\underline{E}$. In the case of $\neg\neg : \Omega \longrightarrow \Omega$, it is easy to verify that $\Omega_{\neg\neg}$ is Boolean, hence $\neg\neg$-Sheaves($\underline{E}$) is the canonical Boolean topos associated to $\underline{E}$ - we will need this later.

The important fact about a is that it is left exact, i.e., preserves finite limits. To see this, we will consider the morphisms inverted by a. Namely, considering the canonical $X \longrightarrow a(X)$ above as the typical sort of morphism inverted by a, we let $\sum$ be the collection of all morphisms $s : X \longrightarrow Y$ such that in the dissection

$$X \overset{\Delta}{\rightarrowtail} X \underset{Y}{\times} X \rightrightarrows X \overset{s}{\longrightarrow} Y, \qquad X \twoheadrightarrow Q \overset{i}{\rightarrowtail} Y \qquad ,$$

both Δ and i are dense. Such s we call <u>bidense</u>. Then we prove $\sum$ has a calculus of right fractions. Namely,

(i.) $X \xrightarrow{id} X$ is in $\sum$ for all X.

(ii.) If $X \xrightarrow{s} Y \xrightarrow{s'} Z$ with $s, s' \in \sum$ then $s' \cdot s \in \sum$.

(iii.) If in the pullback

$$\begin{array}{ccc} f^{*}X & \longrightarrow & X \\ \downarrow & & \downarrow s \\ Z & \xrightarrow{f} & Y \end{array} \quad ,$$

f is arbitrary and $s \in \sum$, then $f^{*}X \longrightarrow Z$ is in $\sum$.

(iv.) If in

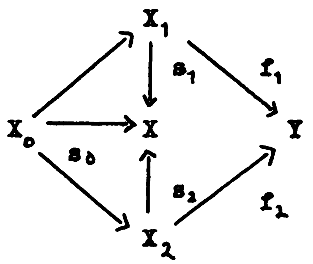
$$Z' \xrightarrow{s'} Z \underset{q}{\overset{f}{\rightrightarrows}} X \xrightarrow{s} Y$$

we have $sf = sq$ with $s \in \sum$, then there exists $s' \in \sum$ with $fs' = qs'$.

This enables us to define $\underline{E}\left[\sum^{-1}\right]$ in the following way. Its objects are the objects of $\underline{E}$, and a map $f : X \longrightarrow Y$ is an equivalence class of pairs

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y \\ s \downarrow & & \\ X & & \end{array} \quad ,$$

where $s \in \sum$ and f' is a morphism of $\underline{E}$. If we write $f = f'|s$ for the equivalence class, then the equivalence relation is $f_1|s_1 = f_2|s_2$ iff there is a commutative diagram

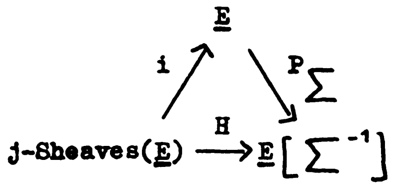


with $s_0 \in \Sigma$. Composition is obvious - by pullback. We have

$$P_\Sigma : \underline{E} \longrightarrow \underline{E}[\Sigma^{-1}]$$

by $P_\Sigma(X) = X$ and $P_\Sigma(f) = f|id$. Since $\underline{E}$ has finite limits and Σ has a calculus of right fractions, it is well known that $\underline{E}[\Sigma^{-1}]$ has finite limits and P_Σ is left exact. For details see [2].

Now consider the diagram



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where $H = P_\Sigma \cdot i$. Then H is full and faithful, and the following are equivalent:

(i.) H is an equivalence.

(ii.) P_Σ has a right adjoint.

(iii.) For all $X \in \underline{E}$ there is a sheaf Y and a map $X \xrightarrow{s} Y$ with $s \in \underline{\Sigma}$.

Since we have verified (iii.), the associated sheaf functor a is left exact.

Thus, to a topology j we have associated an exact reflective subcategory

$$j\text{-Sheaves}(\underline{E}) \underset{i}{\overset{a}{\rightleftarrows}} \underline{E} \qquad .$$

In fact, this correspondence is bijective. Namely, if

$$\underline{E}' \underset{f_*}{\overset{f^*}{\rightleftarrows}} \underline{E}$$

is an exact reflective subcategory of $\underline{E}$ - i.e., $f^* \dashv f_*$, f^* is left exact, and $f^* f_* \xrightarrow{\approx} \text{id}$ - then there is a unique topology j in $\underline{E}$ such that

$$\begin{array}{ccc} & \underline{E} & \\ {\scriptstyle f^*} \swarrow \nearrow {\scriptstyle f_*} & & {\scriptstyle i} \nwarrow \searrow {\scriptstyle a} \\ \underline{E}' & \simeq & j\text{-Sheaves}(\underline{E}) \end{array} \qquad .$$

j is easy to write down; it is

$$\Omega \xrightarrow{\eta(\Omega)} f_* f^*(\Omega) \xrightarrow{\varphi} \Omega \qquad ,$$

where $\eta(\Omega)$ is the adjunction map and φ is the characteristic map of $f_* f^*(\text{true}) : 1 \rightarrowtail f_* f^*(\Omega)$.

5. Morphisms (geometric)

In the previous lecture we started with a topology $j : \Omega \longrightarrow \Omega$ in $\underline{E}$, and constructed an adjoint pair

$$\text{j-Sheaves}(\underline{E}) \underset{i}{\overset{a}{\rightleftarrows}} \underline{E}$$

with $a \dashv i$, a left exact, and i full and faithful. This is a special case of the notion of a geometric morphism of topos. Namely, a geometric morphism $\underline{E}' \xrightarrow{f} \underline{E}$ is an adjoint pair

$$\underline{E}' \underset{f_*}{\overset{f^*}{\rightleftarrows}} \underline{E}$$

with $f^* \dashv f_*$ and f^* left exact. The functors f_* and f^* are called direct image and inverse image, respectifely. The terminology comes from the case of sheaves on a topological space. Namely, a continuous map $f : T \longrightarrow S$ induces a geometric morphism

$$\text{Sheaves}(T) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \text{Sheaves}(S)$$

in the following way. If $f^{-1} : \underline{S} \longrightarrow \underline{T}$ is the obvious functor, then $f_*(F) = F \circ f^{-1}$. Also, if G is a sheaf on S, and $X_G \longrightarrow S$ is the éspace étalé associated to G, then $f^*(G)$ is the sheaf of sections of the éspace étalé $f^*(X_G) \longrightarrow T$ in the pullback

$$\begin{array}{ccc} f^*(X_G) & \longrightarrow & X_G \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & S \end{array} \quad .$$

Here are some examples:

(1.) There is at most one morphism $f : \underline{E} \longrightarrow \underline{S}$, for if $X \in \underline{E}$ then $f_*(X)$ can be identified with its elements; i.e.,

$$f_*(X) \simeq \begin{array}{c} 1 \longrightarrow f_*(X) \\ \hline f^*(1) \longrightarrow X \\ \hline 1 \longrightarrow X \end{array} \quad .$$

Usually we write $\Gamma(X)$ for the set of global sections $1 \longrightarrow X$. If there exists a morphism $\underline{E} \longrightarrow \underline{S}$ we say $\underline{E}$ is defined over $\underline{S}$. More generally, if $f : \underline{E} \longrightarrow \underline{E}_0$ is a morphism we say $\underline{E}$ is defined over $\underline{E}_0$ (relative to f).

In this terminology, $\underline{E}$ is an $\underline{S}$-topos - i.e., a Grothendieck topos relative to $\underline{S}$ - iff $\underline{E}$ is defined over $\underline{S}$ and has an object of generators indexed by an object of $\underline{S}$.

(2.) In the opposite direction, a morphism $\underline{S} \longrightarrow \underline{E}$ is called a <u>point of $\underline{E}$</u>. In the case, say, that T is Hausdorff, the points of Sheaves(T) correspond bijectively to the points of T.

Using topological intuition, which is very good for geometric morphisms, we call a morphism $\underline{E}' \xrightarrow{f} \underline{E}$ <u>surjective</u> if f^* is faithful, and an <u>embedding</u> if f_* is full and faithful. f is said to be <u>essential</u> if there exists a left adjoint $f_!$ for f^*. Essential morphisms correspond to open maps of topological spaces. As an example of how the topological intuition works, we would expect an essential embedding to be an open set, and this is correct. Namely, if $\underline{E}' \xrightarrow{f} \underline{E}$ is an essential embedding, then there is a $U \rightarrowtail 1$ in $\underline{E}$ (these U's are called open for the obvious reason) such that

$$\underline{E}' \quad \begin{array}{c} \xrightarrow{\;f_!\;} \\ \xleftarrow{\;f^*\;} \\ \xrightarrow[\;f_*\;]{} \end{array} \quad \underline{E}$$

is equivalent to

$$\underline{E}/U \quad \begin{array}{c} \xrightarrow{\;\sum_U\;} \\ \xleftarrow{\;(\ )\times U\;} \\ \xrightarrow[\;\prod_U\;]{} \end{array} \quad \underline{E} \qquad .$$

Or, one would expect to be able to factor geometric morphisms into a surjection followed by an embedding, corresponding to the image factorization for continuous maps. This is indeed possible, as we will see.

Kan extensions give another example of essential morphisms. That is, if $\underline{C}$ and $\underline{B}$ are in Cat($\underline{S}$), and if $F : \underline{C} \longrightarrow \underline{B}$ is a functor, then at the level of functor categories we get an essential morphism given by the Kan extensions

$$\underline{S}^{\underline{C}} \quad \begin{array}{c} \xrightarrow{\;\varinjlim_F(\ )\;} \\ \xleftarrow{\;F(\ )\;} \\ \xrightarrow[\;\varprojlim_F(\ )\;]{} \end{array} \quad \underline{S}^{\underline{B}} \qquad .$$

In the special case $\underline{B} = \mathbb{1}$ these become

$$\underline{S}^{\underline{C}} \quad \begin{array}{c} \xrightarrow{\;\varinjlim_{\underline{C}}(\ )\;} \\ \xleftarrow{\;\Delta\;} \\ \xrightarrow[\;\varprojlim_{\underline{C}}\;]{} \end{array} \quad \underline{S} \qquad ,$$

so that the existence of this essential morphism for each F is equivalent to the statement that $\underline{S}$ is complete and cocomplete - i.e., $\underline{S}$-complete and cocomplete. We will see that an arbitrary topos $\underline{E}$ is complete and cocomplete in precisely the same sense.

In order to sketch the proofs of some of the previous statements about geometric morphism, I would like to indicate how it is that surjective morphisms are the same as left exact cotriples. Recall that a <u>cotriple</u> $\mathbf{G} = (G, \varepsilon, \delta)$ on $\underline{E}$ consists of a functor $G : \underline{E} \longrightarrow \underline{E}$ together with natural transformations $\varepsilon : G \longrightarrow \text{id}$ and $\delta : G \longrightarrow G^2$. These should make the diagrams

$$\begin{array}{ccc} G & \xrightarrow{\delta} & G^2 \\ & \searrow^{\text{id}} & \varepsilon G \downarrow \ \downarrow G\varepsilon \\ & & G \end{array} \qquad \& \qquad \begin{array}{ccc} G & \xrightarrow{\delta} & G^2 \\ \delta \downarrow & & \downarrow \delta G \\ G^2 & \xrightarrow[G\delta]{} & G^3 \end{array}$$

commute. Since we will see some examples in a moment, I won't give any now. A <u>coalgebra</u> for $\mathbf{G}$ is a pair (X, θ) where $\theta : X \longrightarrow GX$ satisfies

$$\begin{array}{ccc} X & \xrightarrow{\theta} & GX \\ & \searrow^{\text{id}} & \downarrow \varepsilon X \\ & & X \end{array} \qquad \& \qquad \begin{array}{ccc} X & \xrightarrow{\theta} & GX \\ \theta \downarrow & & \downarrow G\theta \\ GX & \xrightarrow[\delta X]{} & G^2X \end{array}$$

A morphism $(X, \theta) \longrightarrow (Y, \alpha)$ is a map $f : X \longrightarrow Y$ in $\underline{E}$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \theta \downarrow & & \downarrow \alpha \\ GX & \xrightarrow{Gf} & GY \end{array}$$

commutes. Call $\underline{E}_{G}$ the category of coalgebras. Then we have $\underline{E}_{G} \underset{L}{\overset{R}{\rightleftarrows}} \underline{E}$: $L(X, \theta) = X$ and $R(Y) = (GY, \delta Y)$. L is clearly faithful, and it is easy to see that $L \dashv R$. (For more details, and many examples, see [8].) What I claim is: if G is left exact then $\underline{E}_{G}$ is a topos, and $\underline{E} \underset{R}{\overset{L}{\rightleftarrows}} \underline{E}_{G}$ is a surjective morphism. Well, $L : \underline{E}_{G} \longrightarrow \underline{E}$ always creates colimits, so $\underline{E}_{G}$ has finite colimits. Since G is left exact, it follows that L also creates finite limits, so that these are also present in $\underline{E}_{G}$. What about Ω ? Since L creates finite limits, it preserves them, so it preserves monomorphisms. Thus, if $(X, \theta) \rightarrowtail (Y, \alpha)$ then $X \rightarrowtail Y$ so we have a map

$$\frac{Y \longrightarrow \Omega}{(Y, \alpha) \longrightarrow R(\Omega)} \quad .$$

Thus, we must pick out that part of $R(\Omega) = G(\Omega)$ that represents coalgebra monomorphisms - i.e., subobjects that carry a coalgebra structure (there can be at most one). We do this as follows: applying G to $1 \xrightarrow{\text{true}} \Omega$ yields a monomorphism $1 = G(1) \xrightarrow{G(\text{true})} G(\Omega)$; call its characteristic map $\psi : G(\Omega) \longrightarrow \Omega$; then $\Omega_{\mathbb{G}}$ is defined by the equalizer

$$\begin{array}{ccccc} & & 1 & & \\ & \swarrow^{\text{true}_{\mathbb{G}}} & \downarrow{\scriptstyle G(\text{true})} & & \\ \Omega_{\mathbb{G}} & \rightarrowtail & G(\Omega) & \xrightarrow{\text{id}} & G(\Omega) \\ & & {\scriptstyle \delta(\Omega)} \searrow & & \nearrow {\scriptstyle G(\psi)} \\ & & & G^2(\Omega) & \end{array}$$

Note that $\Omega_{\mathbb{G}}$ is automatically a coalgebra. There is a clever way to get exponentiation in $\underline{E}_{\mathbb{G}}$, but let us settle here for a direct formula. Namely, if (X, θ) and (Y, α) are coalgebras, then $(Y, \alpha)^{(X, \theta)}$ is defined by the equalizer

$$\begin{array}{ccccc} (Y, \alpha)^{(X, \theta)} & \rightarrowtail & G(Y^X) & \xrightarrow{G(\alpha^X)} & G(GY^X) \\ & & {\scriptstyle \delta(Y^X)} \downarrow & & \uparrow {\scriptstyle G(GY^\theta)} \\ & & GG(Y^X) & \xrightarrow{G(\nu)} & G(GY^{GX}) \end{array} ,$$

where $\nu : G(Y^X) \longrightarrow GY^{GX}$ corresponds to $G(Y^X) \times GX$

$= G(Y^X \times X) \xrightarrow{G(ev)} GY.$

Let us consider some applications of this result. First, recall that for $\underline{C} \in \mathrm{Cat}(\underline{S})$, we showed that $\underline{S}^{\underline{C}}$ (actually $\underline{S}^{\underline{C}^{op}}$) was a topos. For several reasons, we want to allow $\underline{S}$ to be an arbitrary topos $\underline{E}$ in this construction. Thus, suppose $\underline{C} \in \mathrm{Cat}(\underline{E})$. $\underline{C}$ is given by a diagram

$$I \xleftarrow{\ d\ } M \xrightarrow{\ r\ } I$$

in $\underline{E}$ - M is the object of morphisms of $\underline{C}$, I is the object of objects of $\underline{C}$, d and r are the domain and range maps. Furthermore, we must have a composition law defined on the appropriate pullback and subject to the usual conditions. If this is a category in $\underline{E}$, what is an internal $\underline{E}$-valued functor on $\underline{C}$? Well, these are object assignments $F \longrightarrow I$ provided with an action

$$\begin{array}{ccc} M \times F & \xrightarrow{\ \xi\ } & F \\ {\scriptstyle d}\searrow & & \swarrow \\ & M \xrightarrow{\ r\ } I & \end{array}$$

that satisfies the usual identities. The correct way to say this is:

$$\underline{E}/_I \xrightarrow{d^*} \underline{E}/_M \xrightarrow{\sum_r} \underline{E}/_I$$

is the functor part of a triple $\mathbb{T}$ on $\underline{E}/_I$, and

$$\underline{E}^{\underline{C}} = (\underline{E}/_I)^{\mathbb{T}} \quad ,$$

the category of $\mathbb{T}$ algebras in $\underline{E}/_I$. If $\underline{E} = \underline{S}$, this is precisely $\underline{S}^{\underline{C}}$. Now, in a topos, $\mathbb{T}$ has a right adjoint cotriple $\mathbb{G}$ whose functor part is

$$\underline{E}/_I \xrightarrow{r^*} \underline{E}/_M \xrightarrow{\prod_r} \underline{E}/_I \quad .$$

It is a standard fact that in any such situation one has canonically

$$(\underline{E}/_I)^{\mathbb{T}} \simeq (\underline{E}/_I)_{\mathbb{G}} \quad .$$

Hence, since G is left exact, $\underline{E}^{\underline{C}}$ is a topos, and

$$\underline{E}/_I \;\underset{R}{\overset{L}{\underset{U}{\rightleftarrows}}}\; \underline{E}^{\underline{C}}$$

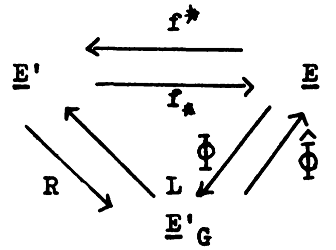
is an essential, surjective morphism. (U here is the L of the previous construction, L is the free algebra functor.) Moreover, suppose F: $\underline{C} \longrightarrow \underline{C}'$ is an $\underline{E}$-functor between $\underline{E}$-categories. F is given by a pair of maps $F_M : M \longrightarrow M'$ and $F_I : I \longrightarrow I'$ that is compatible with domain, range, composition, etc. Then we have the essential pair

$$\underline{E}/I \; \underset{\prod_{F_I}}{\overset{\sum_{F_I}}{\underset{\longleftarrow\, F_I^* \,}{\longrightarrow}}\!\!\longrightarrow} \; \underline{E}/I' \qquad ,$$

and F_I^* lifts to

$$F(\ ) : \underline{E}^{\underline{C}'} \longrightarrow \underline{E}^{\underline{C}}$$

on functors. A standard theorem for triples provides a left adjoint $\underrightarrow{\lim}_F(\ )$ for $F(\ )$, and its dual provides a right adjoint $\underleftarrow{\lim}_F(\ )$. Thus, an arbitrary topos $\underline{E}$ is $\underline{E}$-complete and cocomplete in the same sense that $\underline{S}$ is $\underline{S}$-complete and cocomplete. For a further application, let $f : \underline{E}' \longrightarrow \underline{E}$ be an arbitrary geometric morphism. Then $f^* f_*$ is the functor part of a left exact cotriple $\mathbb{G}$ on $\underline{E}'$, so we can form



where Φ is the standard comparison functor - left exact because f^* is - and $\hat{\Phi}$ is its full and faithful right adjoint given by Beck's cotripleability theorem [8] . There are several alternate ways of describing $\underline{E}'_{\mathbb{G}}$; it is the category of fractions with respect to the morphisms inverted by f^*, or the algebra for the idempotent reflection of the triple $f_* f^*$, or the sheaves for the topology induced by f (see the end of lecture 4). In any case, we can factor geometric morphism uniquely as a surjection followed by an embedding, which is what we wanted to prove.

6. Morphisms (logical)

Quite a different sort of morphism is a logical morphism of topos. These are functors

$$\underline{E}' \xrightarrow{1} \underline{E}$$

that preserve the complete topos structure - i.e., finite limits and colimits, exponentiation, and Ω. As an example, let $X \in \underline{E}$, where $\underline{E}$ is a topos. Then in

$$\underline{E}/X \quad \begin{array}{c} \xrightarrow{\sum_X} \\ \xleftarrow{(\ ) \times X} \\ \xrightarrow[\prod_X]{} \end{array} \quad \underline{E} \qquad ,$$

( ) x X is logical. In fact, this is an example of what we call a <u>local</u> <u>homeomorphism</u>, i.e., a geometric morphism $f : \underline{E}' \longrightarrow \underline{E}$ such that f^* is logical. Speaking very roughly, logical morphisms arise as a result of preforming logical constructions in a topos rather than geometric ones. Such constructions lead us, for the first time, out of the realm of Grothendieck topos. That is, these are constructions, which, when preformed on a Grothendieck topos, will yield a topos, but not a Grothendieck topos. Thus, as we will see shortly, our topos can be used to construct, say, non-standard models of set theory, whereas this would be impossible with Grothendieck topos.

Our principal example of such a construction is a sort of ultrapower for topos. Let us consider this in some detail, as we will need it later. Thus, let $\underline{E}$ be a topos, and let $\mathcal{F}$ be a filter (external) of subobjects of 1. That is, if $U \rightarrowtail 1$ is in $\mathcal{F}$ and if $U < V$, then $V \in \mathcal{F}$, and if U and V are in $\mathcal{F}$ so is $U \wedge V$. Define a class Σ of monomorphisms of $\underline{E}$ by putting $X' \rightarrowtail X$ in Σ iff $\prod_X X' \in \mathcal{F}$. Then if

$$\rho : \underline{E} \longrightarrow \underline{E}[\Sigma^{-1}]$$

denotes the canonical projection to the category of fractions, we claim that $\underline{E}[\Sigma^{-1}]$ is a topos and ρ is logical.

Let us establish first that Σ has a calculus of right fractions, and its saturation $\overline{\Sigma}$ has a calculus of left fractions. This will provide finite limits and colimits in $\underline{E}[\Sigma^{-1}]$, and show they are preserved by ρ. Well, for any $X \in \underline{E}$ certainly $id_X \in \Sigma$, and in any diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{s} Z \qquad ,$$

if $sf = sg$ for $s \in \sum$, then $f = g$, i.e., $fid_X = gid_X$ so a pair of maps coequalized by a member of $\sum$ is equalized by a member of $\sum$. Thus, for right fractions we are left with pullbacks and composites. However, these are easily taken care of with the following facts. First, for a pullback diagram

$$\begin{array}{ccc} f^*Y' & \longrightarrow & Y' \\ {\scriptstyle s'}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

we have $\prod_Y Y' < \prod_X f^*Y'$, so if $s \in \sum$, so is s'. Then for composites $X'' \overset{s'}{\rightarrowtail} X' \overset{s}{\rightarrowtail} X$, we have $(\prod_{X'} X'') \wedge (\prod_X X') < \prod_X X''$, so if s' and s are in $\sum$ so is ss'. By right fractions it follows at once that if $X \xrightarrow{f} Y$ and $\rho(f)$ is an isomorphism then it has a right inverse in $\underline{E}[\sum^{-1}]$, which yields a diagram

$$\begin{array}{ccc} X' & \overset{i}{\rightarrowtail} & X \\ & {\scriptstyle s}\searrow & \big\downarrow{\scriptstyle f} \\ & & Y \end{array}$$

in $\underline{E}$, with $s \in \sum$. Note that if f is monic, then $f \in \sum$. Then, however, the equation $\rho(f) \cdot \rho(i) = \rho(s)$ shows that $\rho(i)$ is an isomorphism, hence $i \in \sum$.

Thus, the saturation $\overline{\overline{\Sigma}}$ of Σ consists precisely of those $X \xrightarrow{f} Y$ for which there exists a diagram of the above form in $\underline{E}$ with i and s in Σ. Since composites and identities are trivially in $\overline{\overline{\Sigma}}$, we must consider coequalizers and pushouts. First, suppose $X \xrightarrow{f} Y$ is in $\overline{\overline{\Sigma}}$. Consider the dissection

$$\begin{array}{ccccccc} X & \overset{\Delta}{\rightarrowtail} & X \underset{Y}{\times} X & \overset{\pi_2}{\underset{\pi_1}{\rightrightarrows}} & X & \xrightarrow{f} & Y \\ & & & & & {\scriptstyle q}\searrow\!\!\!\!\searrow \quad \nearrow{\scriptstyle i} & \\ & & & & & Q & \end{array} \quad .$$

Then, if $\rho(f)$ is an isomorphism $\rho(i)$ is split epic, but since ρ preserves monomorphisms it follows that $i \in \Sigma$. Also, $\rho(f)$ monic yields $\rho(\pi_1) = \rho(\pi_2)$, hence $\Delta \in \Sigma$. Thus, for coequalizers it is enough to show the following : in the diagram,

$$X' \overset{s}{\rightarrowtail} X \overset{f}{\underset{g}{\rightrightarrows}} Y \overset{q}{\twoheadrightarrow} Q \quad ,$$

if $q = \text{coeq}(f, g)$ where $fs = gs$ with $s \in \Sigma$, then $q \in \overline{\Sigma}$. For this, cross the diagram with $\prod_X X'$. It remains a coequalizer, but also, we have the adjunction

$$\begin{array}{ccc} (\prod_X X') \times X & \xrightarrow{\ \varepsilon\ } & X' \\ & \pi_1 \searrow \quad \swarrow s & \\ & X & \end{array}$$

Thus,

$$(\prod_X X') \times X \xrightarrow{\langle \pi_1 , \varepsilon \rangle} (\prod_X X') \times X') \xrightarrow[X]{(\prod X')\times s} (\prod_X X') \times X$$

is the identity, hence

$$(\prod_X X') \times f = (\prod_X X') \times g \quad ,$$

making $(\prod_X X') \times q$ an isomorphism. But then $q \in \Sigma$, since in

$$\begin{array}{ccc} (\prod X') \times Y & \xrightarrow[X]{(\prod X') \times q} & (\prod_X X') \times Q \\ \pi_2 \downarrow & & \downarrow \pi_2 \\ Y & \xrightarrow{\ q\ } & Q \end{array}$$

both π_2 's are in Σ . Now consider a pushout

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{g'} & P \end{array} \quad .$$

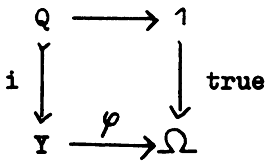
If $f \in \overline{\overline{\Sigma}}$ we want to know that $f' \in \overline{\overline{\Sigma}}$. It is enough to know this for $f \in \Sigma$, since the above remark and the result for coequalizers will take care of the epic part of f. But now it is easy to show that $\prod_Y X < \prod_P Z$, which completes the proof.

For exponentiation, one checks first that $\underline{E}[\Sigma^{-1}]$ is cartesian closed and ρ preserves exponentiation iff for $Y' \overset{s}{\rightarrowtail} Y$, $s \in \Sigma$ yields $s^X \in \Sigma$ for all $X \in \underline{E}$. The latter condition holds, however, since it is easy to show that $\prod_Y Y' < \prod_{Y^X} Y'^X$.

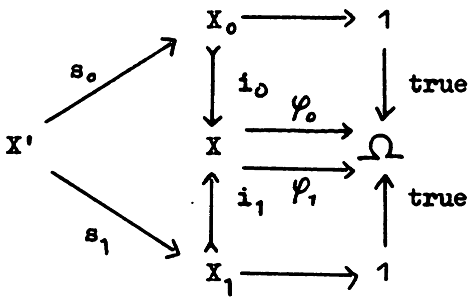
Finally, let us show that $\rho : \underline{E} \longrightarrow \underline{E}[\Sigma^{-1}]$ preserves Ω. That is, $1 \xrightarrow{\rho(\text{true})} \rho(\Omega)$ classifies subobjects in $\underline{E}[\Sigma^{-1}]$. First of all, in $\underline{E}[\Sigma^{-1}]$ an arbitrary map $f|s = \rho(f) \cdot \rho(s)^{-1}$ is monic iff $\rho(f)$ is monic, iff in the factorization

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle q}\searrow \quad \nearrow{\scriptstyle i} & \\ & Q & \end{array}$$

$\rho(q)$ is an isomorphism. But i has a characteristic map $\varphi : Q \longrightarrow \Omega$, and



remains a pullback upon application of ρ, so that $\rho(f)$ is obtained as the pullback of ρ(true) along $\rho(\varphi)$. Thus, our problem reduces to the question of the uniqueness of the characteristic map, and this, in turn, reduces to the following special case. Namely, suppose we have a diagram



,

where the top and bottom squares are pullbacks, and s_0, $s_1 \in \Sigma$. Then the diagrams

$$
\begin{array}{ccc}
X' & \rightarrowtail & \prod_{i_0} X' \\
{\scriptstyle s_0}\downarrow & & \downarrow \\
X_0 & \xrightarrow{\ i_0\ } & X
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
X' & \rightarrowtail & \prod_{i_1} X' \\
{\scriptstyle s_1}\downarrow & & \downarrow \\
X_1 & \xrightarrow{\ i_1\ } & X
\end{array}
$$

are both pullbacks, and their right hand vertical edges are in $\sum$. Thus, we have

$$
\begin{array}{ccccc}
 & & \prod_{i_0} X' & & \\
 & & \downarrow & & \\
X' \rightarrowtail (\prod_{i_0} X') \wedge (\prod_{i_1} X') & \xrightarrow{\ \bar{s}\ } & X & \overset{\varphi_0}{\underset{\varphi_1}{\rightrightarrows}} & \Omega , \\
 & & \uparrow & & \\
 & & \prod_{i_1} X' & &
\end{array}
$$

with $\bar{s} \in \sum$ and $\varphi_0 \bar{s} = \varphi_1 \bar{s}$. The latter equation holds since $\varphi_0 \bar{s}$ and $\varphi_1 \bar{s}$ have the same pullback of true, namely

$$
X' \rightarrowtail (\prod_{i_0} X') \wedge (\prod_{i_1} X') \quad .
$$

Since we will need it later, we should also mention that if $\underline{E}$ has a natural number object ω , then $\rho(\omega)$ is a natural number object for $\underline{E}[\sum^{-1}]$.

The proof of this is straightforward, and we omit it.

Here is an alternate description of $\underline{E} \xrightarrow{\rho} \underline{E}[\Sigma^{-1}]$. Consider the diagram

$$U \rightsquigarrow \underline{E}/_U$$

for $U \in \mathfrak{F}$, where if $U < V$ then $\underline{E}/_V \longrightarrow \underline{E}/_U$ by restriction. Then $\underline{E} \xrightarrow{\rho} \underline{E}[\Sigma^{-1}]$ is canonically equivalent to $\underline{E} \longrightarrow \varinjlim_{U \in \mathfrak{F}} \underline{E}/_U$ - the injection of the direct limit (taken in $\underline{\text{Cat}}$) corresponding to $1 \in \mathfrak{F}$. One proves this by showing that $\underline{E} \longrightarrow \varinjlim_{U \in \mathfrak{F}} \underline{E}/_U$ has the same universal property as ρ, which is immediate, since $\underline{E}/_U$ is the category of fractions for the class of monomorphisms $X' \rightarrowtail X$ such that $X' \times U \xrightarrow{\simeq} X \times U$, and this is the same as the class of monomorphisms $X' \rightarrowtail X$ such that $U < \prod_X X'$.

A common way to obtain such filters on $\underline{E}$ is to assume given a finite limit preserving functor $f : \underline{E} \longrightarrow \underline{E}_o$, together with a homomorphism $h : f(\Omega) \longrightarrow \Omega_o$ of Heyting algebras in $\underline{E}_o$. One obtains then a filter $\mathfrak{F}$ on $\underline{E}$ by letting $U \rightarrowtail 1$ be in $\mathfrak{F}$ iff the composite

$$1 \xrightarrow{f(\varphi_U)} f(\Omega) \xrightarrow{h} \Omega_o$$

is true$_\circ$, where φ_U is the characteristic map of U. A special case should illustrate what this construction has to do with the classical theory of ultrapowers. Namely, for $I \in \underline{E}$ consider the local homeomorphism

$$\underline{E}/I \underset{\prod_I}{\overset{(\ )\times I}{\rightleftarrows}} \underline{E}$$

First of all, notice that $\underline{E}/I$ is really "$\underline{E}$ to the power I" (consider the case $\underline{E} = \underline{S}$). Then, as the Ω for $\underline{E}/I$ is $\Omega \times I \xrightarrow{\pi_2} I$, we are assuming given a Heyting algebra homomorphism

$$h : \prod_I (\Omega \times I) \simeq \Omega^I \longrightarrow \Omega ,$$

that is, an ultrafilter on I if $\underline{E} = \underline{S}$. Now the terminal object 1 in $\underline{E}/I$ is $I \xrightarrow{id} I$, so the filter $\mathcal{F}$ on $\underline{E}/I$ consists of subobjects

$$\begin{array}{ccc} I' & \rightarrowtail & I \\ & \searrow \quad \swarrow_{id} & \\ & I & \end{array}$$

such that if we take their characteristic maps - i.e.,

$$\begin{array}{ccc} I & \xrightarrow{\langle \varphi,\, id\rangle} & \Omega \times I \\ & {\scriptstyle id}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & I & \end{array}$$

where φ is the characteristic map of $I' \rightarrowtail I$ - apply $\prod_I$ giving

$$1 \xrightarrow{\prod_I \langle \varphi\, ,\, id\rangle} \prod_I (\Omega \times I) \simeq \Omega^I \ ,$$

then h of this is true. However, it is easy to see that under the isomorphism $\prod_I (\Omega \times I) \simeq \Omega^I$, $\prod_I \langle \varphi\, ,\, id\rangle$ is identified with $1 \xrightarrow{\overline{\varphi}} \Omega^I$. Thus $\mathfrak{J}$ consists of the "elements" of the "ultrafilter" on I. What are the monomorphisms of $\sum$? Well, these are diagrams

$$\begin{array}{ccc} X' & \rightarrowtail & X \\ & \searrow \quad \swarrow{\scriptstyle f} & \\ & I & \end{array}$$

such that $\prod_f X' \rightarrowtail I$ is in $\mathfrak{J}$. But

$$\prod_f X' = \text{"}\{i \in I \mid \text{`}X'_i = X_i\}\text{"} \quad .$$

Thus, the monomorphisms we invert are those for which this "set" is in the "ultrafilter". Now, since ( ) x I is logical, we have a logical morphism

$$\underline{E} \xrightarrow{(\ ) \times I} \underline{E}/I \xrightarrow{\rho} \underline{E}/I[\Sigma^{-1}] .$$

In fact, if Ω has no non-trivial Heyting algebra endomorphisms (e.g., if $\underline{E}$ is Boolean), then the above morphism is an elementary embedding, which for topos means a logical morphism that reflects isomorphisms. Thus, we obtain the classical theorem when $\underline{E} = \underline{S}$.

7. The Continuum Hypothesis

In this final lecture I would like to use the constructions we have developed to study briefly the Continuum Hypothesis in set theory. (For complete details the reader should consult [10].) As a beginning, we must make precise what we mean by a model of set theory. The axioms we will use are, as a group, equivalent to the axioms for the Elementary Theory of the Category of Sets [5], but the emphasis is quite different. Out point of view here is that the category of sets is merely a special sort of topos. In [5], for example, one could prove the independence of the axiom of choice by considering the category of partially ordered sets, which is a model of all the axioms except AC. With the new axioms this will not work, for the category of partially ordered sets is far from being a topos.

Thus, by a <u>model of set theory</u> we mean, first, a Boolean topos $\underline{S}$ with a natural number object ω . Next, $\underline{S}$ should satisfy the axiom of choice: any epimorphism $q : X \twoheadrightarrow Y$ has a section $s : Y \longrightarrow X$, such that $qs = id_Y$. So far, this is what we would call a model for Boolean set theory. To pin down classical set theory as 2-valued Boolean set theory, we require: if $U \rightarrowtail 1$ then $U \simeq 0$ or $U \simeq 1$.

If we call this system CS - for the category of sets - a natural question is: What does CS have to do with ZF? For the answer, I will merely refer the reader to [1] or [6], except to say that they are essentially equivalent. That is, they are equivalent if you add a suitable form of replacement. We will not discuss replacement, since it plays no role in our arguments, but one can formulate replacement in an arbitrary topos, and its study should be interesting.

The Continuum Hypothesis (CH) is a categorical question that can be posed in the following way. Does there exist an X such that $\omega \rightarrowtail X \rightarrowtail 2^{\omega}$, but $\omega \not\cong X \not\cong 2^{\omega}$? To show that there are models of set theory in which such an X exists, we must construct new models from our given $\underline{S}$, and this is where our previous constructions come in.

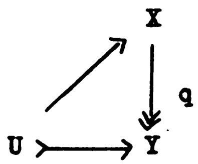
So, calling the objects of $\underline{S}$ sets, let P be a partially ordered set in $\underline{S}$. Then $\underline{P}$ (formed as in lecture 1) is a category - of a particularly simple kind - in $\underline{S}$, so we can use the functor category construction to build $\underline{S}^{\underline{P}}$, which, as we have seen, is a topos. Unless $\underline{P}$ is discrete, however, $\underline{S}^{\underline{P}}$ is not Boolean, so we pass to its Booleanization - the $\neg\neg$-sheaves in $\underline{S}^{\underline{P}}$. Let us call this $Sh_{\neg\neg}(\underline{P})$. Then we have adjoints

$$Sh_{\neg\neg}(\underline{P}) \underset{i}{\overset{a}{\rightleftarrows}} \underline{S}^{\underline{P}} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \underline{S}$$

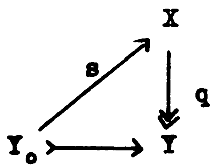
We write $\wedge$ for $a\Delta$, and simply Γ for Γ i. Thus, $Sh_{\neg\neg}(\underline{P})$ is defined over $\underline{S}$ by

$$Sh_{\neg\neg}(\underline{P}) \underset{\Gamma}{\overset{\wedge}{\rightleftarrows}} \underline{S}$$

Now $Sh_{\neg\neg}(\underline{P})$ is a Boolean topos with natural number object $\omega = \hat{\omega}$. It also satisfies AC. Indeed, the subobjects of 1 form a system of generators for $\underline{S}^{\underline{P}}$, so the same is true of $Sh_{\neg\neg}(\underline{P})$. Let $q : X \twoheadrightarrow Y$ be an epimorphism. If $Y \neq 0$, then $X \neq 0$ and there is a map $U \longrightarrow X$ with $U \rightarrowtail 1$ and $U \neq 0$ - otherwise $0 \longrightarrow X$ is an isomorphism. Thus q has a partial section:



Now consider the partially ordered set of all such partial sections. It is non-empty by the above, and inductive since $Sh_{\neg\neg}(\underline{P})$ is a topos. Using Zorn's Lemma in $\underline{S}$, let



be a maximal element. Then $Y_o \neq Y$ iff $Y - Y_o \neq 0$. Forming the pullback

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle q'}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ Y - Y_0 & \rightarrowtail & Y \end{array}$$

q' is epic, so that if $Y - Y_0 \neq 0$ it has a partial section with non-zero domain. Then, however, we have properly extended s and contradicted its maximality. So $Y_0 = Y$ and AC holds. Thus, we have a model for Boolean set theory.

To get a 2-valued model we use ultrapowers. Namely, by the maximal ideal theorem, choose a Boolean homomorphism $h : \Gamma(\Omega_{\neg\neg}) \longrightarrow 2$ in $\underline{S}$, and put

$$\underline{S}(\underline{P}, h) = Sh_{\neg\neg}(\underline{P})[\Sigma^{-1}]$$

Then $\underline{S}(\underline{P}, h)$ is a Boolean topos with a natural number object ω, and

$$\rho : Sh_{\neg\neg}(\underline{P}) \longrightarrow \underline{S}(\underline{P}, h)$$

is logical. For AC and 2-valuedness, see [10]. Thus, we have many new models of set theory, and their properties will depend on $\underline{P}$ (and h to a lesser extent).

Now, how shall we negate CH? Well, in any topos $\underline{E}$, if $X, Y \in \underline{E}$ we can define $\mathrm{Epi}(X, Y) \rightarrowtail Y^X$ - the "object of epimorphisms from X to Y" (see [10] again for details). Its global sections correspond to the actual apimorphisms from X to Y, though, of course, it has other sections as well. Having Epi(X, Y), we choose P in $\underline{S}$ so that in $Sh_{\neg\neg}(\underline{P})$ there is an X with

$$\omega \rightarrowtail X \rightarrowtail 2^{\omega}$$

-recall $\Omega_{\neg\neg} \simeq 1 + 1 = 2$ - but

$$\mathrm{Epi}(\omega, X) = \mathrm{Epi}(X, 2^{\omega}) = 0$$

Once we have done this we are done, for such an internal negation remains negated upon application of $\rho : Sh_{\neg\neg}(\underline{P}) \longrightarrow \underline{S}(\underline{P}, h)$.

The final step, then, is to choose P. Here we proceed in a standard manner. Namely, using Cantor's diagonal argument, we first choose a set I such that $2^{\omega} \rightarrowtail I$, but $\mathrm{Epi}(2^{\omega}, I) = 0$. (Note that in a model of set theory, the latter statement has its usual meaning, i.e., there exist no epimorphisms from 2^{ω} to I.)

Now let P be the set of partial maps from $I \times \omega$ to 2 with finite domain. Thus, a $p \in \underline{P}$ is a partial map

$$\begin{array}{ccc} F & \longrightarrow & 2 \\ \big\downarrow & & \\ I \times \omega & & \end{array}$$

where domain $p = F$ is finite, and $F \longrightarrow 2$ is arbitrary. Put $p \leq q$ iff $\operatorname{dom} p \subseteq \operatorname{dom} q$ and $q \mid \operatorname{dom} p = p$. One of the main reasons for choosing $\underline{P}$ to be of this form - i.e., partial maps with finite domain and codomain, regardless of the size of I - is that when we pass to $Sh_{\neg\neg}(\underline{P})$, the Boolean algebra $\Gamma(\Omega_{\neg\neg})$ satisfies the countable chain condition; any set of pairwise disjoint elements is countable. It is this condition that enables us to prove the fundamental result: If $X \neq 0$ and $X \simeq X \times \omega$, then for any Y, $\operatorname{Epi}(X, Y) = 0$ in $\underline{S}$ yields $\operatorname{Epi}(\hat{X}, \hat{Y}) = 0$ in $Sh_{\neg\neg}(\underline{P})$.

The next step, which is the essence of forcing, is to produce a monomorphism $\hat{I} \rightarrowtail 2^{\omega}$ in $Sh_{\neg\neg}(\underline{P})$. Recalling that $2 \simeq \Omega_{\neg\neg}$ and $\omega = \hat{\omega}$, what we want is a map $\hat{I} \longrightarrow \Omega_{\neg\neg}^{\hat{\omega}}$, or $\hat{I} \times \hat{\omega} \longrightarrow \Omega_{\neg\neg}$, or $\widehat{I \times \omega} \longrightarrow \Omega_{\neg\neg}$, or $I \times \omega \longrightarrow \Gamma(\Omega_{\neg\neg})$, and we define this by sending $\langle i, n\rangle$ into $\{p \mid p\langle i, n\rangle = \text{true}\}$ - this latter set is a $\neg\neg$-closed filter of elements of P, i.e., a global section of $\Omega_{\neg\neg}$. Again, I refer to [10] for the proof that the resulting map $\hat{I} \longrightarrow 2^{\omega}$ is monic.

Granting these two results we are finished. For we have $\omega \rightarrowtail 2^{\omega} \rightarrowtail I$ in $\underline{S}$, giving

$$\omega \rightarrowtail \widehat{2^{\omega}} \rightarrowtail \hat{I} \rightarrowtail 2^{\omega}$$

in $Sh_{\neg\neg}(\underline{P})$. Moreover, since $\mathrm{Epi}(\omega, 2^{\omega}) = \mathrm{Epi}(2^{\omega}, I) = 0$ in $\underline{S}$, we have $\mathrm{Epi}(\omega, \widehat{2^{\omega}}) = \mathrm{Epi}(\widehat{2^{\omega}}, \hat{I}) = 0$ in $Sh_{\neg\neg}(\underline{P})$. But $\hat{I}$ has global sections, onto any one of which we may map its complement in 2^{ω}, so there exists a retraction $r : 2^{\omega} \twoheadrightarrow \hat{I}$. r induces a map $\mathrm{Epi}(\widehat{2^{\omega}}, 2^{\omega}) \longrightarrow \mathrm{Epi}(\widehat{2^{\omega}}, \hat{I})$ by composition, so $\mathrm{Epi}(\widehat{2^{\omega}}, 2^{\omega}) = 0$, and the X we are seeking is $\widehat{2^{\omega}}$.

Thus, we see that the elementary nature of our topos allow applications denied to Grothendieck topos. Broadly speaking, it seems to be the case that any geometric construction that can be preformed on a Grothendieck topos can be preformed on ours - often the methods are even simpler; see [9], for more examples - but our topos are also stable under several new logical constructions that Grothendieck topos are not.

With regard to this application, we should also point out that this model building technique admits of considerable extension. That is, we can consider arbitrary categories of forcing conditions, not just partially ordered sets; in addition, we can choose different topologies to suit different needs. Not much has been done with this to date, but the prospect seems hopeful.

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