SHEAF THEORY AND THE CONTINUUM HYPOTHESIS

by

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In this paper I would like to give an account of some joint work of myself and F. W. Lawvere which is concerned with establishing the independence of the Continuum Hypothesis from other axioms of the category of sets. More precisely, we will show that the Continuum Hypothesis does not follow from the axioms of the Elementary Theory of the Category of Sets [3]. Some discussion of the relationship of this result to that of Cohen can be found at the end of the paper.

The exposition will involve several results about topos - i.e. categories of sheaves - and these will be simply stated and used without proof. The reason for this is that, properly speaking, the material of this paper is an application of our axiomatic theory of sheaves - described in [4] and [7]-and should really be so presented. Thus, although we define the concept of topos below, we make no attempt to develop the theory here, since to do so only to prove this result would be to put the cart before the horse.

Since the axioms we use here for the category of sets are somewhat different from those of [3], though equivalent as a group, we should begin by discussing these in some detail. The first, unnumbered, group states that the universe of discourse is a category

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$\underline{S}$. The next group, which defines a topos in the sense of [4] is concerned with those properties of sets that can be described by adjointness. Namely, we assume:

Axiom 1. All finite limits and colimits exist.

As is well-known, to satisfy Axiom 1 it is enough to have an initial object 0 , a terminal object 1, a pullback for every pair of maps with common codomain, and a pushout for every pair of maps with common domain.

Axiom 2. $\underline{S}$ is cartesian closed.

That is, for all X and Y there is an object of maps X^Y with the universal property of λ-conversion. Namely,

$$\frac{Z \longrightarrow X^Y}{Z \times Y \longrightarrow X}$$

with the obvious assumptions of naturality. More compactly, though somewhat improperly, we can express this by saying that for all Y , the functor $(\ ) \times Y$ has a right adjoint $(\ )^Y$.

Axiom 3. Subobjects in $\underline{S}$ are representable.

Precisely, there is an object Ω together with a map $1 \longrightarrow \Omega$, which is called "true," such that for any monomorphism $X' \rightarrowtail X$ (such arrows will always denote monomorphisms) there is a unique characteristic map $\phi : X \longrightarrow \Omega$ such that

$$\begin{array}{ccc} X' & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow \text{true} \\ X & \xrightarrow{\ \phi\ } & \Omega \end{array}$$

is a pullback.

A category $\underline{E}$ satisfying Axioms 1-3 will be called a topos. If $\underline{S}$ is a category of sets, then certainly 1-3 should hold, where X^Y is the set of all maps from Y to X, and Ω is the 2-point set. More generally, suppose $\underline{C}$ is a category in $\underline{S}$, and consider $\underline{S}^{\underline{C}}$ the category of covariant $\underline{S}$-valued functors on $\underline{C}$. Certainly Axiom 1 holds, since limits and colimits in $\underline{S}^{\underline{C}}$ are computed pointwise. If C is an object of $\underline{C}$, let us denote by the same letter the representable functor $(C,-)$ in $\underline{S}^{\underline{C}}$. Then the Yoneda lemma says that for any $F \in \underline{S}^{\underline{C}}$, the natural transformations from C to F are in natural 1-1 correspondence with the set $F(C)$. We use this to determine exponentation and Ω in $\underline{S}^{\underline{C}}$. Namely, if F and G are functors, and if F^G is to exist at all, then we must have

$$\frac{C \longrightarrow F^G}{C \times G \longrightarrow F} .$$

Using this as a definition, one checks that it works. Similarly, if Ω is to exist, then in particular,

$$\frac{C \longrightarrow \Omega}{R \rightarrowtail C}$$

i.e., the value of Ω at C must be the collection of subobjects (in $\underline{S}^{\underline{C}}$) of the representable functor C — these are called cribles — and again one sees that this works. Thus $\underline{S}^{\underline{C}}$ is a topos. In some sense, the characteristic example of a topos is obtained by taking T to be a topological space in $\underline{S}$, and forming Sheaves (T) — the category of $\underline{S}$-valued sheaves on T. Finite limits and colimits are again clear, though one must be a little more careful with the colimits. Since the representable functors in the category of presheaves — i.e., the open sets of T — are themselves sheaves, the same reasoning as in the previous example shows the existence of exponentiation and Ω. Much of the terminology used in the theory of topos comes from this special case. For example, if $U \rightarrowtail 1$ we call U <u>open</u>, a map $1 \longrightarrow X$ is called a <u>global section</u> of X, etc.

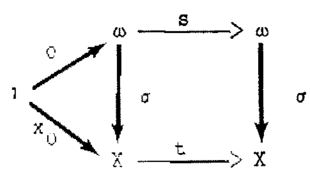
In general, if $\underline{E}$ and $\underline{E}_0$ are topos, a <u>geometric morphism</u> of topos $f:\underline{E} \longrightarrow \underline{E}_0$ will be an adjoint pair

$$\underline{E} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \underline{E}_0 \quad ,$$

in which f^* is left adjoint to f_* and preserves finite limits If there exists a geometric morphism from $\underline{E}$ to $\underline{E}_0$, we say $\underline{E}$ is <u>defined over</u> $\underline{E}_0$.

The next axiom, which asserts the existence of a natural number object, is somewhat different from the previous three in that, though it deals with a property of sets expressible in terms of the existence of a certain adjoint, if $\underline{E}$ is defined over $\underline{E}_0$ and $\underline{E}_0$ satisfies the axiom, so does $\underline{E}$.

<u>Axiom 4</u>. (Axiom of infinity). There is an object ω, together with maps $1 \xrightarrow{0} \omega \xrightarrow{s} \omega$, such that for any object X provided with maps $1 \xrightarrow{x_0} X \xrightarrow{t} X$ there exists a unique map $\sigma:\omega \longrightarrow X$ such that



commutes.

Though we will not pursue the point here (see [4]), Axioms 1-3 are sufficient for the complete, axiomatic, development of sheaf theory. That is, we can deduce from them the typical exactness properties of set-valued sheaf categories, define topologies, pass to sheaves, etc. One exactness property that we will need later, and might as well state now, is the following: In any category $\underline{E}$

with pullbacks, a map $f:X \longrightarrow Y$ induces a functor

$$\underline{E}/Y \xrightarrow{f^*} \underline{E}/X$$

given by pulling an object over Y back along f to an object over X. It is always true that f^* has a left adjoint Σ_f — just compose with f — but in a topos f^* also has a right adjoint Π_f. This is a much stronger property — it implies, for example, that f^* preserves colimits.

As an example of how one operates with Ω in a topos, let us see what kind of algebraic structure Ω must carry. We already have the map $1 \xrightarrow{\text{true}} \Omega$, and after establishing the fact that $0 \rightarrowtail 1$ is monic, we obtain — as its characteristic map — the map $1 \xrightarrow{\text{false}} \Omega$. Next, there is an operation

$$\wedge : \Omega \times \Omega \longrightarrow \Omega$$

called conjunction, which, by the universal property of Ω, is completely defined by specifying where it takes the value "true." As everyone knows, this is precisely at the pair $\langle \text{true}, \text{true} \rangle$, hence $\wedge$ is the characteristic map of the subobject determined by

$$1 \xrightarrow{\langle \text{true}, \text{true} \rangle} \Omega \times \Omega \quad .$$

Similarly, disjunction, written

$$\vee : \Omega \times \Omega \longrightarrow \Omega \quad ,$$

is the characteristic map of

$$(\Omega \times 1) \cup (1 \times \Omega) \rightarrowtail \Omega \times \Omega \quad .$$

Now we can define the order relation $\Omega_0 \rightarrowtail \Omega \times \Omega$ as an equalizer

$$\Omega_0 \rightarrowtail \Omega \times \Omega \underset{\wedge}{\overset{\pi_1}{\rightrightarrows}} \Omega \quad ,$$

and its characteristic map gives the operation

$$\Longrightarrow : \Omega \times \Omega \longrightarrow \Omega \quad ,$$

which we call implication. With respect to these operations, one proves that Ω is a Heyting algebra object in $\underline{E}$. For category theorists, this means that Ω is a trivial category object (by means of the ordering defined above) which is finitely complete and cocomplete and cartesian closed ($\Longrightarrow$ is the exponentiation for the product $\wedge$). Continuing in this vein we can define negation, written

$$\neg : \Omega \longrightarrow \Omega \quad ,$$

as the characteristic map of $1 \xrightarrow{\text{false}} \Omega$. Underscoring the intuitionistic character of topos we have the fact that

$$\neg\neg : \Omega \longrightarrow \Omega$$

is rarely the identity. For example, if T is a T_0-space, then in Sheaves (T), $\neg\neg = id$ iff T is discrete. However, we will certainly want our set theory to be classical, and this forms the content of the next axiom. From among several possible ways of stating this we choose the following: $1 \xrightarrow{\text{false}} \Omega$ and $1 \xrightarrow{\text{true}} \Omega$ Provide a map $1 + 1 \longrightarrow \Omega$, and we require

Axiom 5. $1 + 1 \longrightarrow \Omega$ is an isomorphism.

Other conditions we might have used are: Ω is a Boolean algebra, $\neg\neg = id$, subobjects have complements, etc. We call a topos Boolean if it satisfies Axiom 5.

We shall further require the axiom of choice, which we state as

Axiom 6. Epimorphisms split.

That is, for every epimorphism $q:X \twoheadrightarrow Q$ (such arrows always denote epimorphisms) there exists a map $s:Q \longrightarrow X$ such that $qs = id$. s is called a section for q .

Axioms 1 - 6 describe Boolean set theory - with the axiom of choice-and perhaps a remark should be made about that. Namely, for X in an arbitrary topos $\underline{E}$, factoring the canonical map $X \longrightarrow 1$

yields

$$\begin{array}{ccc} X & \longrightarrow & 1 \\ & \searrow \quad \nearrow & \\ & \sigma(X) & \end{array}$$

where we call $\sigma(X)$ the support of X. If the epi part of the above factorization always splits, we say that supports split in $\underline{E}$. Now if $\underline{E}$ is Boolean, and supports split, then the subobjects of 1 form a system of generators for $\underline{E}$, which is a kind of extensionality condition. Thus, even when investigating Boolean set theory without AC one should still require that supports split.

Finally, although we have required Ω to be $1 + 1$, it can still be very large — in fact its global sections can be an arbitrary complete Boolean algebra. Hence, to pin down the category of sets as 2-valued set theory, we require

Axiom 7. If $U \rightarrowtail 1$, then $U \simeq 0$ or $U \simeq 1$.

Before going on, it is probably worthwhile to make a few remarks on this system, which we might call CS — for the category of sets. As indicated previously, Axioms 1 - 7, as a group, are equivalent to the axioms given in [3], though the emphasis here is placed quite differently. Roughly speaking, the point of view here is that the category of sets is merely one among many topos, and one should learn not to specialize too soon. In [3], for example, much more weight was placed on the axiom of choice, whose independence could be shown by considering the category of partially ordered sets, this being a model of all the axioms except AC. Here this will never work, since the category of partially ordered sets is far from being even a topos, which is as it should be.

The extent to which CS describes, say, the category of sets built from ZF will be touched on at the end of the paper. Suffice to say here that, though the axiom of replacement is lacking, one seems to be able to develop most of the mathematics one would like to in CS . We might say a few words about uniqueness, however. Namely, suppose $\underline{S}$ is a model for CS. Then we can prove that topos defined over $\underline{S}$, in which the subjobjects of 1 generate , are in $1 - 1$ correspondence with complete Heyting algebras in $\underline{S}$. In particular, then, any other model of CS defined over $\underline{S}$ is equivalent to $\underline{S}$. This is the precise form of the statement that if you add the non-elementary axiom of completeness, then you have characterized set theory.

Now, for a moment, let us discuss the Continuum Hypothesis. This is a categorical question which we can phrase as follows: Does there exist an X such that

$$\omega \rightarrowtail X \rightarrowtail 2^{\omega}$$

properly? That is, $X \not\cong \omega$ and $X \not\cong 2^{\omega}$. What we shall do is to start with a model $\underline{S}$ of CS and construct another model $\underline{S}'$ in which such an X exists. Very briefly, the procedure is this: By considering functors on partially ordered sets in $\underline{S}$, we can find many models for Axioms 1-4 . By passing to sheaves in these "presheaf" categories, we can pick up Axioms 5 and 6. By choosing a particular partially ordered set we can negate CH in the internal topos logic, which is much stronger than merely negating it in the external form above (though equivalent in the 2-valued case). Now, collapsing along a morphism that preserves the topos structure we can find a 2-valued model in which CH is false. Let us begin with the general construction of models for Boolean set theory. In our fixed model $\underline{S}$ of CS, we will use an ε relation defined as in [3]. Although our use of it here is somewhat informal, the dilegent reader should have no trouble supplying any desired details.

Call the objects of $\underline{S}$ sets, and let $\mathbb{P}$ be a partially ordered set. Then $\mathbb{P}$ is a trivial category via its ordering — i.e., $p \longrightarrow q$ iff $p \leq q$ — and we can form the category $\underline{S}^{\mathbb{P}}$ of covariant $\underline{S}$-valued functors on $\mathbb{P}$. These functors should be thought of as sets parametrized by $\mathbb{P}$. That is, $X \in \underline{S}^{\mathbb{P}}$ is given by specifying its value at $p \in \mathbb{P}$; $X(p) \in \underline{S}$; and requiring for $p \leq q$ that $X(p) \longrightarrow X(q)$ in a transitive way. $\mathbb{P}$ itself is embedded in $\underline{S}^{\mathbb{P}}$ by the Yoneda functor which sends $p \in \mathbb{P}$ into the representable functor defined by p (also written simply as p) — i.e., $p \in \underline{S}^{\mathbb{P}}$ is given by

$$p(q) = \begin{cases} 1 & \text{if } p \leq q \\ 0 & \text{otherwise} \end{cases} .$$

Since $\underline{S}^{\mathbb{P}}$ is a special case of the construction $\underline{S}^{\underline{C}}$ discussed earlier, $\underline{S}^{\mathbb{P}}$ is a topos. Moreover, $\underline{S}^{\mathbb{P}}$ is defined over $\underline{S}$ by means of the adjoint pair

$$\underline{S}^{\mathbb{P}} \underset{(1,-)}{\overset{\Delta}{\rightleftarrows}} \underline{S}$$

where $(1,-)$ at $X \in \underline{S}^{\mathbb{P}}$ is the set of maps — in $\underline{S}^{\mathbb{P}}$ — from 1 to X , and $\Delta S(p) = S$ for all $p \in \mathbb{P}$. Therefore, Axiom 4 holds in $\underline{S}^{\mathbb{P}}$. Recall that $\Omega \in \underline{S}^{\mathbb{P}}$ is the functor whose value at $p \in \mathbb{P}$ is the collection of subobjects — in $\underline{S}^{\mathbb{P}}$ — of the representable functor p . If $p \leq q$, then $\Omega(p) \longrightarrow \Omega(q)$ is given by pulling back the subobject $R \rightarrowtail p$ along the map $q \longrightarrow p$. Now such a subobject is completely determined by specifying those $q \geq p$ at which R takes the value 1, so we will identify subobjects of p with filters of elements of $\mathbb{P}$ that are $\geq p$ — i.e., sets of elements R of $\mathbb{P}$ that are $\geq p$ and have the property that if $q \in R$ and $q' \geq q$, then $q' \in R$. The map $1 \xrightarrow{\text{true}} \Omega$ picks out, for every $p \in \mathbb{P}$, the filter of all elements $\geq p$, i.e., p itself, and $1 \xrightarrow{\text{false}} \Omega$ picks out the empty filter for each p.

Now, although $\underline{S}^{\mathbb{P}}$ is a topos, it will not be Boolean unless $\mathbb{P}$ is discrete, so we must go further in order to obtain a Boolean topos. Since the deviation of $\neg\neg : \Omega \longrightarrow \Omega$ from the identity measures the failure of $\underline{S}^{\mathbb{P}}$ to be Boolean, let us calculate this map explicitly. Since $\neg : \Omega \longrightarrow \Omega$ is the characteristic map of $1 \xrightarrow{\text{false}} \Omega$, it follows that if $X' \rightarrowtail X$ has characteristic map $\phi : X \longrightarrow \Omega$, then $\neg X' \rightarrowtail X$ — the subobject corresponding to $\neg\phi$ — is the subfunctor given by

$$\neg X'(p) = \{x \in X(p) \mid \phi(p)(x) = 0\} \quad .$$

If $q \geq p$, then $X(p) \longrightarrow X(q)$, and writing x_q for the value of this map on $x \in X(p)$, we have

$$\neg X'(p) = \{x \in X(p) \mid \forall\, q \geq p,\ x_q \notin X'(q)\} \quad .$$

To illustrate the correspondence of subobjects with characteristic maps, let us prove this. So suppose $x \in X(p)$ is such that $\forall q \geq p,\ x_q \notin X'(q)$ but $\phi(p)(x) = R \rightarrowtail p$ is not 0 . Then for some $q \geq p$, $R(q) = 1$, i.e.,

$$\begin{array}{ccc} q & \longrightarrow & R \\ \| & & \downarrow \\ q & \longrightarrow & p \end{array}$$

is a pullback. But then, since the diagram

$$\begin{array}{ccccc} X'(p) & \rightarrowtail & X(p) & \xrightarrow{\phi(p)} & \Omega(p) \\ \downarrow & & \downarrow & & \downarrow \\ X'(q) & \rightarrowtail & X(q) & \xrightarrow{\phi(q)} & \Omega(q) \end{array}$$

commutes, we have $\phi(q)(x_q) = q$, or $x_q \in X'(q)$. The reverse implication is similar. Applying $\neg$ we see that

$\neg\neg X'(p) = \{x \in X(p) \mid \forall q \geq p \exists r \geq q \text{ with } x_r \in X'(r)\}$.

Clearly, $X' \rightarrowtail \neg\neg X'$. One calls $X' \rightarrowtail X$ dense if $\neg\neg X' = X$, and closed if $\neg\neg X' = X'$. By the above claculations, $X' \rightarrowtail X$ is dense iff for any $p \in \mathbb{P}$ and $X \in X(p)$ there exists $r \geq p$ with $x_r \in X'(r)$, and closed iff, given $x \in X(p)$ such that $\forall q \geq p \exists r \geq q$ with $x_r \in X'(r)$, it follows that $x \in X'(p)$.

If we identify subobjects $R \rightarrowtail p$ with filters of elements of $\mathbb{P} \geq p$, and use the Yoneda correspondence between maps $p \longrightarrow \Omega$ and elements of $\Omega(p)$ we find that if $R \rightarrowtail p$, then

$$\neg R = \{q \geq p \mid \forall q' \geq q,\ q' \notin R\}$$

and

$$\neg\neg R = \{q \geq p \mid \forall q' \geq q \exists r \geq q' \text{ with } r \in R\} \quad .$$

Thus, $R \rightarrowtail p$ is dense iff $\forall q \geq p \exists r \geq q$ such that $r \in R$, and closed iff, given $q \geq p$ such that $\forall q' \geq q \exists r \geq q$ with $r \in R$, then $q \in R$. Note that if $\mathbb{P}$ has a maximal element, then any nonempty crible is dense.

In a moment we shall show that the collection of dense subobjects of p, for each $p \in \mathbb{P}$, forms a Grothendieck topology on $\mathbb{P}$ called the $\neg\neg$-topology. Before doing this, however, let us remark that in the general treatment it is $\neg\neg$ itself that is the topology, and it is present in any topos. In a topos of the form $\underline{S}^{\underline{C}}$, a topology given by an endomorphism of Ω satisfying certain axioms is equivalent to a Grothendieck topology on $\underline{C}$, but the former makes sense in any topos and is idependent of the notion of "site." For now, however, let us simply verify the Grothendieck-

Verdier axioms [2] for a topology:

(i) For $p \in \mathbb{P}$, $p \xrightarrow{id} p$ is dense (clear)

(ii) If $q \geq p$ — i.e., $q \longrightarrow p$ in $\underline{S}^{\mathbb{P}}$ — and $R \rightarrowtail p$ is dense, so is $R|q \rightarrowtail q$ (clear).

(iii) Suppose $R' \rightarrowtail p$, and $R \rightarrowtail p$ is a dense crible with the property that $\forall q \in R$, $R'|q \rightarrowtail q$ is dense. We must show $R' \rightarrowtail p$ is dense. So, suppose $q \geq p$. Then, since $\exists r \geq q$ such that $r \in R$, $R'|r \rightarrowtail r$ is dense. In particular, $\exists r' \geq r$ with $r' \in R'$ and so $R' \rightarrowtail p$ is dense.

Now, having a topology we may consider separated objects, sheaves, etc. Again we remark that this process of passing to sheaves can — and should — be carried out in an arbitrary topos. Here, however, we will say $X \in \underline{S}^{\mathbb{P}}$ is <u>separated</u> iff for any $p \in \mathbb{P}$ and any dense $R \rightarrowtail p$, the canonical map

$$X(p) \longrightarrow \varprojlim_{q \in R} X(q)$$

is monic. X is called a $\neg\neg$-<u>sheaf</u> if the above map is an isomorphism. As reinforcement for one's topological intiution, one readily checks that X is separated iff the diagonal $X \rightarrowtail X \times X$ is closed. In the same vein, X is a sheaf iff it is separated and <u>absolutely closed</u>, i.e., closed in any separated object containing it. Notice, for later reference, that if $S \in \underline{S}$, then ΔS — the constant presheaf at S — is certainly separated, but need not be a sheaf.

Passing to sheaves, let

$$Sh_{\neg\neg}(P) \xrightarrow[i]{} \underline{S}^{\mathbb{P}}$$

denote the inclusion of the full subcategory of sheaves for the $\neg\neg$-topology. From [2], or better, [4], we take the following facts: There is an associated sheaf functor

$$a: \underline{S}^{\mathbb{P}} \longrightarrow Sh_{\neg\neg}(\mathbb{P})$$

which is left adjoint to i and preserves finite limits. Moreover, if $X \in \underline{S}^{\mathbb{P}}$ is separated, then the unit $X \longrightarrow ia(X)$ is monic and dense. Composing the adjoint pairs

$$Sh_{\neg\neg}(\mathbb{P}) \underset{i}{\overset{a}{\rightleftarrows}} \underline{S}^{\mathbb{P}} \underset{(1,-)}{\overset{\Delta}{\rightleftarrows}} \underline{S} \quad ,$$

we obtain a single pair

$$Sh_{\neg\neg}(\mathbb{P}) \underset{\Gamma}{\overset{\wedge}{\rightleftarrows}} \underline{S} \quad .$$

Thus, if X is a $\neg\neg$-sheaf $\Gamma(X)$ = set of global sections of X, and if $S \in \underline{S}$, then $\hat{S}$ is the sheaf associated to the constant presheaf at S.

Here we pause for a moment to prove a technical lemma that will be useful later on. Namely, suppose $S \in \underline{S}$. Then we have

<u>Lemma 1</u>. For any section $p \xrightarrow{x} \hat{S}$, there is an element $s \in S$ and an $r \geq p$ such that

$$\begin{array}{ccccc} & & p & & \\ & \nearrow & & \searrow^{x} & \\ r & & & & \hat{S} \\ & \searrow & & \nearrow_{\hat{s}} & \\ & & 1 & & \end{array}$$

commutes.

Proof: As above, we have a dense monomorphism $\Delta S \rightarrowtail \hat{S}$. Thus, as was established previously, for any $p \in \mathbb{P}$ and $x \in \hat{S}(p)$, $\exists r \geq p$ such that $x_r \in \Delta S(r)$. By the Yoneda lemma, this is the same as saying that for any map $p \xrightarrow{x} \hat{S}$, $\exists r \geq p$ such that $r \longrightarrow p \xrightarrow{x} \hat{S}$ factors through $\Delta S \rightarrowtail S$ — i.e., we have a diagram

$$\begin{array}{ccc} r & \longrightarrow & p \\ \vdots\downarrow & & \downarrow x \\ \Delta S & \rightarrowtail & \hat{S} \end{array} .$$

$\Delta S(r) = S$, so identify $r \longrightarrow \Delta S$ with an element $s \in S$. Thinking of s as a map $1 \xrightarrow{s} S$ we have by naturality a commutative diagram

$$\begin{array}{ccc} \Delta 1 & \xrightarrow{\sim} & \hat{1} \\ {\scriptstyle \Delta s}\downarrow & & \downarrow{\scriptstyle \hat{s}} \\ \Delta S & \rightarrowtail & \hat{S} \end{array} .$$

Now composing on the right with the canonical map $r \longrightarrow \Delta 1$ gives the result.

Notice that the assertion of the lemma is the analogue of the statement for sheaves on a topological space that sections of a constant sheaf are locally constant functions into the fibre.

Our first theorem is

Theorem 1. $Sh_{\neg\neg}(\mathbb{P})$ is a model of Boolean set theory.

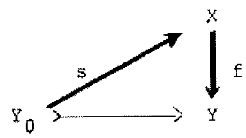
Proof: We must establish Axioms 1-6 in $Sh_{\neg\neg}(\mathbb{P})$. Well, Axiom 1 is trivial, and Axiom 2 holds since it is easy to verify that if $X \in Sh_{\neg\neg}(\mathbb{P})$ and Y is any presheaf, then X^Y is a sheaf. Also, Axiom 4 is trivial, since $Sh_{\neg\neg}(\mathbb{P})$ is defined over $\underline{S}$.

What about the subobject classifier of Axiom 3? First of all, in any topos — in particular $\underline{S}^{\mathbb{P}}$ — Ω is injective, so if we define $\Omega_{\neg\neg}$ as an equalizer

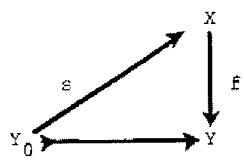
$$\Omega_{\neg\neg} \rightarrowtail \Omega \underset{id}{\overset{\neg\neg}{\rightrightarrows}} \Omega \quad ,$$

it follows that $\Omega_{\neg\neg}$ is also injective. To prove that $\Omega_{\neg\neg}$ is a sheaf, then, we need only show that it is separated. So, suppose $R \rightarrowtail p$ and $R' \rightarrowtail p$ are closed subobjects of p that have the same restriction to each element q of some dense $R_0 \rightarrowtail p$. If $R \neq R'$, then there is a $q \in R$, say, such that $q \notin R'$. R_0 is dense, so that $\forall q' \geq q \exists r \in R_0$ with $r \geq q'$. $r \geq q'$, and $q \in R$, so $r \in R$. But $R|r = R'|r$, hence $r \in R'|r$, i.e., $r \in R'$. Since $R' \rightarrowtail p$ is closed, $q \in R'$, which is a contradiction. Thus $R = R'$ and $\Omega_{\neg\neg}$ is a sheaf. Since $\Omega_{\neg\neg}$ classifies closed subobjects by definition, it follows that if $X' \rightarrowtail X$ is closed and X is a sheaf, so is X' (the pullback of a sheaf is a sheaf). On the other hand, if X' is a sheaf then $X' \rightarrowtail X$ is closed since X is separated and X' is absolutely closed. All of this adds up to the fact that the Ω for $Sh_{\neg\neg}(\mathbb{P})$ is precisely $\Omega_{\neg\neg}$. Moreover, from a standard Heyting algebra argument we can deduce that $\Omega_{\neg\neg}$ is a Boolean algebra object. Thus $Sh_{\neg\neg}(\mathbb{P})$ is a Boolean topos — i.e., Axiom 5 holds.

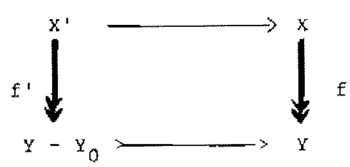
Finally, we establish Axiom 6 in $Sh_{\neg\neg}(\mathbb{P})$ by using the existence of complements for arbitrary subobjects, and Zorn's lemma in $\underline{S}$. To start, let $X \overset{f}{\twoheadrightarrow} Y$ be an epimorphism in $Sh_{\neg\neg}(\mathbb{P})$. We claim that if $Y \neq 0$, then there exists a partial section of f with non-zero domain — i.e., a commutative diagram



with $Y_0 \neq 0$. In fact, this is true for an arbitrary f with $X \neq 0$. Namely, it is easy to see that the subobjects of 1 form a system of generators in $\neg\neg$-$Sh(\mathbb{P})$. Thus, if $X \neq 0$ there is a map $U \longrightarrow X$ with U a non-zero subobject of 1 (otherwise $0 \rightarrowtail X$ is an isomorphism). Now just take $Y_0 \rightarrowtail Y$ to be the composite $U \longrightarrow X \xrightarrow{t} Y$. If f is epic, though, it must actually split globally. To see this, consider the partially ordered set (in $\underline{S}$) of all such partial sections. It is non-empty clearly, and inductive since $Sh_{\neg\neg}(\mathbb{P})$ is a topos. By Axiom 6 in $\underline{S}$, let



be a maximal element. Then $Y_0 \neq Y$ iff $Y - Y_0 \neq 0$. Forming the pullback



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f' is epic, so if $Y - Y_0 \neq 0$ we can find a partial section of f' with non-zero domain, thus properly extending s and contradicting its maximality. Thus $Y_0 = Y$, f splits, and $Sh_{\neg\neg}(\mathbb{P})$ satisfies the axiom of choice.

We discuss next how to retrieve a 2-valued model from $Sh_{\neg\neg}(\mathbb{P})$. First, we remark that $B = \Gamma(\Omega_{\neg\neg})$ is a complete Boolean algebra in $\underline{S}$. Next, using Zorn's lemma, choose a Boolean homomorphism $h: B \longrightarrow 2$. If $X' \rightarrowtail X$ is a monomorphism in $Sh_{\neg\neg}(P)$, let $\Pi_X X' \rightarrowtail 1$ denote the result of applying to $X' \rightarrowtail X$ the right adjoint to pulling back along $X \longrightarrow 1$. Let $\phi_{X'}: 1 \longrightarrow \Omega_{\neg\neg}$ be its characteristic map. Now put

$$\Sigma = \{X' \rightarrowtail X \mid h(\phi_{X'}) = \text{true}\} \quad ,$$

write $\underline{S}(\mathbb{P},h)$ for the category of fractions $Sh_{\neg\neg}(\mathbb{P})\ [\Sigma^{-1}]$, and let

$$P: Sh_{\neg\neg}(\mathbb{P}) \longrightarrow \underline{S}(\mathbb{P},h)$$

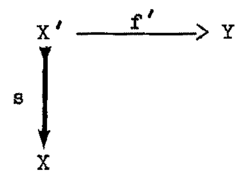
denote the canonical projection. Then we have

Theorem 2. $\underline{S}(\mathbb{P},h)$ is a model of set theory. Moreover, P preserves finite limits, finite colimits, exponentiation, Ω , and ω .

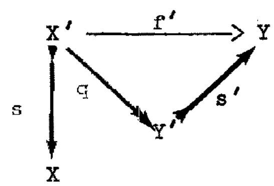
Proof: We sketch only the particular aspects of the proof, since the formation of $\underline{S}(\mathbb{P},h)$ is a special case of a general construction discussed in [4]. That is, we cite [4] for the proof that $\underline{S}(\mathbb{P},h)$ is a topos and P preserves the topos structure. If we call such a functor a logical morphism of topos, then, very roughly, P is a logical morphism because Σ has a calculus of right fractions and is closed under exponentiation and its saturation $\bar{\Sigma}$ has a calculus of left fractions. In any case, it follows that P preserves the statement

$$1 + 1 \xrightarrow[\simeq]{} \Omega \quad ,$$

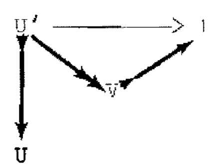
so that $\underline{S}(\mathbb{P},h)$ is also Boolean. What about Axioms 6 and 7 for $\underline{S}(\mathbb{P},h)$? Well, recall from [1], that a map $f: X \longrightarrow Y$ in $\underline{S}(\mathbb{P},h)$ can be represented in the form



where $s \in \Sigma$. Now factor f' in $Sh_{\neg\neg}(\mathbb{P})$, giving a diagram



Since P preserves both epimorphisms and monomorphisms (and $\underline{S}(\mathbb{P},h)$ is a topos), it follows that if f is epic, $s' \in \Sigma$, in which case f splits in $\underline{S}(\mathbb{P},h)$ since q splits in $Sh_{\neg\neg}(\mathbb{P})$. Thus the axiom of choice holds in $\underline{S}(\mathbb{P},h)$. Also, if f is monic then $q \in \bar{\Sigma}$ — the saturation of Σ. In particular, suppose f is of the form $U \rightarrowtail 1$ in $\underline{S}(\mathbb{P},h)$. Then the above factorization looks like



with $U' \twoheadrightarrow V$ in $\bar{\Sigma}$. Let $\varphi_V : 1 \longrightarrow \Omega_{\neg\neg}$ be the characteristic map of $V \rightarrowtail 1$ in $Sh_{\neg\neg}(\mathbb{P})$. Then either $h(\varphi_V) = \text{true}$ or $h(\varphi_V) = \text{false}$. If $h(\varphi_V) = \text{true}$ then $V \rightarrowtail 1$ is in Σ, so $U \rightarrowtail 1$ is an isomorphism in $\underline{S}(\mathbb{P},h)$. If $h(\varphi_V) = \text{false}$, then

$h(\neg\varphi_V)$ = true and $\neg V \rightarrowtail 1$ is in Σ. But

$$\begin{array}{ccc} 0 & \rightarrowtail & \neg V \\ \downarrow & & \downarrow \\ V & \rightarrowtail & 1 \end{array}$$

is a pullback in $Sh_{\neg\neg}(\mathbb{P})$, so $0 \rightarrowtail V$ is in Σ. But then U is isomorphic to 0 in $\underline{S}(\mathbb{P},h)$, and $\underline{S}(\mathbb{P},h)$ satisfies Axiom 7.

Provided with Theorems 1 and 2, we can give a more precise idea of how it is one shows CH does not follow from Axioms 1-7. First, given X and Y in any topos $\underline{E}$, we can define the internal object of epimorphisms Epi(X,Y). To do this, start by defining the map "image": $Y^X \longrightarrow \Omega^Y$. To give such a map we need a map $Y^X \times Y \longrightarrow \Omega$ or, equivalently, a subobject of $Y^X \times Y$. For this, take the image $S \rightarrowtail Y^X \times Y$ of the map

$$\begin{array}{ccc} Y^X \times X & \xrightarrow{\langle \pi_1, ev\rangle} & Y^X \times Y \\ & \searrow \quad \nearrow & \\ & S & \end{array},$$

where ev is the evalution map. It is easy to see that if $f:X \longrightarrow Y$, then the composite

$$1 \xrightarrow{\bar{f}} Y^X \longrightarrow \Omega^Y$$

is the transpose of the characteristic map of the image of f. Letting true_Y be the map $Y \longrightarrow 1 \xrightarrow{\text{true}} \Omega$, we define Epi(X,Y) as a pullback

$$\begin{array}{ccc} \text{Epi}(X,Y) & \rightarrowtail & Y^X \\ \downarrow & & \downarrow \text{im} \\ 1 & \xrightarrow{\overline{\text{true}_Y}} & \Omega^Y \end{array}$$

Thus, the global sections of Epi(X,Y) correspond to the actual epimorphisms $X \longrightarrow Y$, but there will also be many sections over other objects. Now the procedure is simple. Namely, we shall choose $\mathbb{P}$ in such a way that in $Sh_{\neg\neg}(\mathbb{P})$ there will exist an X with the property (remember $\Omega_{\neg\neg} \simeq 1 + 1 = 2$)

$$\omega \rightarrowtail X \rightarrowtail 2^{\omega}$$

but $\mathrm{Epi}(\omega,X) = \mathrm{Epi}(X,2^{\omega}) = 0$. Once we do this we are finished, for $P: Sh_{\neg\neg}(\mathbb{P}) \longrightarrow \underline{S}(\mathbb{P},h)$ preserves the topos structure, thus it also preserves such an internal negation of CH.

Now, to pick $\mathbb{P}$ first choose an $I \in \underline{S}$ such that $2^{\omega} \rightarrowtail I$ but $2^{\omega} \not\twoheadrightarrow I$ — this can be done by Cantor's diagonal argument. Then let $\mathbb{P}$ be the collection of partial maps from $I \times \omega$ to 2 with finite domain — i.e., a $p \in \mathbb{P}$ is a partial map

$$\begin{array}{ccc} F & \longrightarrow & 2 \\ \big\downarrow & & \\ I \times \omega & & \end{array}$$

where F is finite and $F \longrightarrow 2$ is arbitrary. Put $p \leq q$ iff $\mathrm{dom}\, p \subset \mathrm{dom}\, q$ and $q|\mathrm{dom}\, p = p$.

One important reason for choosing $\mathbb{P}$ to be a set of partial maps with finite domain (and codomain) is that any such set satisfies the ω-chain condition. More precisely, let T be a finite set with $\mathrm{card}(T) = m$, and let $J \in \underline{S}$. If $\mathbb{Q}$ is the object of partial maps from J to T with finite domain, then $\mathbb{Q}$ has the following property:

Lemma 2. If $Z \rightarrowtail \mathbb{Q}$ is such that no pair q,q' in Z has a common extension in $\mathbb{Q}$ — meaning there exists no p in $\mathbb{Q}$ with $p \geq q$ and $p \geq q'$ — then $\omega \twoheadrightarrow Z$ — i.e., Z is countable.

Proof: We first reformulate the condition on Z. Namely, for $q,q' \in Z$ there exists no p in $\mathbb{Q}$ with $p \geq q$ and $p \geq q'$ iff there is a $j \in \mathrm{dom}(q) \cap \mathrm{dom}(q')$ such that $q(j) \neq q'(j)$. Now suppose that for each $q \in Z$ $\mathrm{card}(\mathrm{dom}(q)) = n$. Then we claim

$$\mathrm{card}(Z) \leq n!m^n \quad ,$$

which we prove by induction on n. The statement being clearly true for $n = 1$, assume it for n and suppose $\mathrm{card}(\mathrm{dom}(q)) = n + 1$ for all $q \in Z$. Then, for $j \in J$ and $t \in T$, let

$$Z(j,t) = \{q \in Z \mid j \in \mathrm{dom}(q) \text{ and } q(j) = t\} \quad .$$

Fixing some $q_0 \in Z$, we have

$$Z = \bigcup_{\substack{j \in \mathrm{dom}(q_0) \\ t \in T}} Z(j,t) \quad .$$

For each $Z(j,t)$, form Z' by removing j from $\mathrm{dom}(q)$ for all $q \in Z(j,t)$. Now if $q \neq q'$ in $Z(j,t)$ then there is a $j' \in \mathrm{dom}(q) \cap \mathrm{dom}(q')$ such that $q(j') \neq q'(j')$. Thus $j' \neq j$, so

$$\mathrm{card}(Z(j,t)) = \mathrm{card}(Z')$$

and Z' still satisfies the original condition on Z. But now, $\mathrm{card}(\mathrm{dom}(q) = n$ for all $q \in Z'$, so

$$\mathrm{card}(Z') \leq n!m^n \quad .$$

But this gives

$$\begin{aligned} \mathrm{card}(Z) &\leq (n+1)mn!m^n \\ &= (n+1)!m^{n+1} \quad . \end{aligned}$$

Now, of course, for an arbitrary Z satisfying the given disjointness condition, if

$$Z_n = \{q \in Z \mid \mathrm{card}(\mathrm{dom}(q)) = n\}$$

$Z = \bigcup_n Z_n$ so Z is countable.

We might remark that, although this result suffices for our purposes, one can obviously use the same proof to get various stronger results. For example, T could be countable, or one could prove a more general cardinality result from the start. Also, we should probably say that we call this the ω-chain condition because if one forms $Sh_{\neg\neg}(\mathbb{Q})$, then the above condition on $\mathbb{Q}$ is equivalent to the usual ω-chain condition on the Boolean algebra $B = \Gamma(\Omega_{\neg\neg})$.

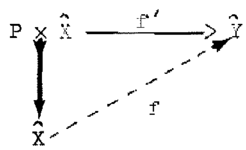
Going back to our original $\mathbb{P}$, consider the adjoint pair

$$Sh_{\neg\neg}(\mathbb{P}) \underset{\Gamma}{\overset{\wedge}{\rightleftarrows}} \underline{S} \quad .$$

The importance of the ω-chain condition is that it enables us to establish one of the two basic results used in negating CH. Namely,

<u>Theorem 3</u>. If $X,Y \in \underline{S}$ and X is infinite (meaning $X \neq 0$ and $X \cong X \times \omega$) then $Epi(X,Y) = 0$ in $\underline{S}$ yields $Epi(\hat{X},\hat{Y}) = 0$ in $Sh_{\neg\neg}(\mathbb{P})$.

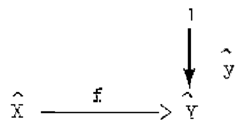
<u>Proof</u>: Suppose $Epi(\hat{X},\hat{Y}) \neq 0$. Then there is a map $p \longrightarrow Epi(\hat{X},\hat{Y})$ for some $p \in \mathbb{P}$. But this, it is easy to see, is the same as a map $f':p \times \hat{X} \longrightarrow \hat{Y}$ such that $\langle \pi_1, f' \rangle : p \times \hat{X} \twoheadrightarrow p \times \hat{Y}$ is epic. By Booleanness one can extend f' to $f:\hat{X} \longrightarrow \hat{Y}$ — i.e., there is a diagram



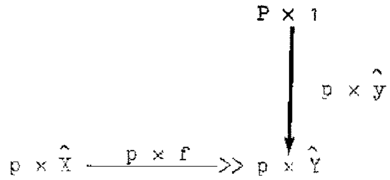
This is so because $\hat{Y}$ has global sections onto any one of which we can map the complement of $p \times \hat{X} \rightarrowtail \hat{X}$. Now $p \times f = \langle \pi_1, f' \rangle$, so $p \times f : p \times \hat{X} \twoheadrightarrow p \times \hat{Y}$ is epic. In $\underline{S}$, define $E \rightarrowtail \mathbb{P} \times X \times Y$ by

$$E = \{(p,x,y) \mid p \rightarrowtail 1 \xrightarrow{\hat{x}} \hat{X} \xrightarrow{f} \hat{Y} \text{ commutes (with } 1 \xrightarrow{\hat{y}} \hat{Y})\}$$

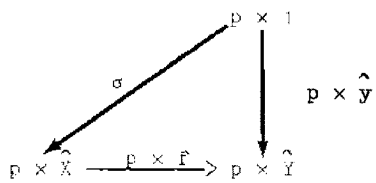
Let $E \longrightarrow \mathbb{P}$, $E \longrightarrow X$, and $E \longrightarrow Y$ be the maps obtained by composing $E \rightarrowtail \mathbb{P} \times X \times Y$ with the various projections. First, we claim $E \longrightarrow Y$ is epic. Well, consider $1 \overset{y}{\rightarrowtail} Y$ and



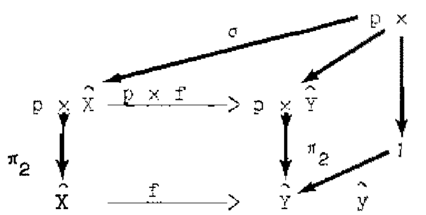
Crossing with p yields



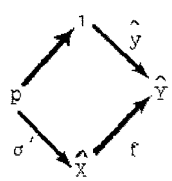
By AC in $Sh_{\neg\neg}(\mathbb{P})$, $p \times f$ splits, so there is a map $\sigma : p \times 1 \longrightarrow p \times X$ such that



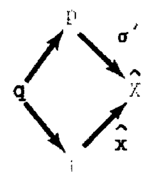
commutes. Composing with π_2 yields a diagram



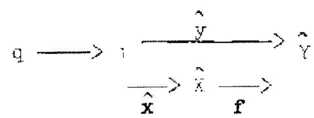
If we write σ' for $\pi_2 \circ \sigma$, then



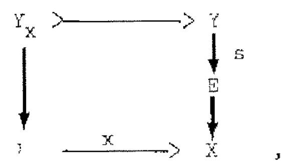
commutes. But then by Lemma 1, $\exists q \geq p$ and $x \in X$ such that



commutes. Thus the two composites in



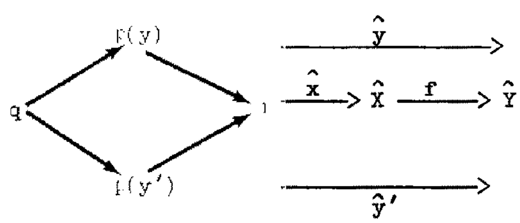
are equal, so $(q,x,y) \in E$. By AC in $\underline{S}$, let $Y \xrightarrow{s} E$ be a section for $E \longrightarrow Y$. If $1 \xrightarrow{x} X$, let Y_x be the pullback



i.e.,

$$Y_x = \{y \in Y \mid s(y) = (q,x,y) \text{ for some } q\} .$$

Denote the composite $Y_x \longrightarrow Y \xrightarrow{s} E \longrightarrow \mathbb{P}$ by $p: Y_x \longrightarrow \mathbb{P}$ — i.e., if $y \in Y_x$ and $s(y) = (q,x,y)$, then $p(y) = q$. Then not only is p monic, but moreover, the condition of Lemma 2 is satisfied. That is, if $y \neq y'$ in Y_x, then there is no $p \in \mathbb{P}$ such that $q \geq p(y)$ and $q \geq p(y')$. For suppose so. Then from the diagram



we see that

$$q \longrightarrow 1 \overset{\hat{y}}{\underset{\hat{y}'}{\rightrightarrows}} \hat{Y}$$

commutes. But in that case, the diagram

$$\begin{array}{ccccc} q & \longrightarrow & \Delta 1 & \overset{y}{\underset{y'}{\rightrightarrows}} & \Delta Y \\ & & \Big\downarrow \rho_1 & & \Big\downarrow \\ & & \hat{1} & \overset{\hat{y}}{\underset{\hat{y}'}{\rightrightarrows}} & \hat{Y} \end{array}$$

yields $y = y'$. Thus, for each $x \in X$ we have an epimorphism $\rho_x : \omega \longrightarrow\!\!\!\!\rightarrow Y_x$ by the ω-chain condition for $\mathbb{P}$. Now define an epimorphism $\rho : X \times \omega \longrightarrow\!\!\!\!\rightarrow Y$ by $\rho(x,n) = \rho_x(n)$. But then, since $X \cong X \times \omega$, we have constructed an element of $\mathrm{Epi}(X,Y)$ in $\underline{S}$, and this is a contradiction.

The next step, which is the essence of forcing, is to prove

<u>Theorem 4</u>. In $Sh_{\neg\neg}(\mathbb{P})$ there is a monomorphism

$$\hat{I} \rightarrowtail \Omega_{\neg\neg}^{\omega} \quad .$$

If we prove this, then since $2 = 1 + 1 \cong \Omega_{\neg\neg}$ in $Sh_{\neg\neg}(P)$, we will have been able to choose an arbitrary degree of largeness $2^{\omega} \rightarrowtail I$ in $\underline{S}$, and force 2^{ω} to be at least as large as $\hat{I}$ in $Sh_{\neg\neg}(\mathbb{P})$.

<u>Proof</u>: We start by giving a map

$$\varphi : \Delta I \times \Delta\omega \longrightarrow \Omega_{\neg\neg}$$

in $\underline{S}^{\mathbb{P}}$, or what is the same thing, a closed subobject

$$R \rightarrowtail \Delta I \times \Delta\omega \quad .$$

To do this, put

$$R(p) = \{\langle i,n\rangle \mid p\langle i,n\rangle = \text{true}\} \quad .$$

Let us check that $R \rightarrowtail \Delta I \times \Delta\omega$ is closed. Well, suppose $\langle i,n\rangle \in (\Delta I \times \Delta\omega)(p)$ is such that $\forall q \geq p\ \exists r \geq q$ with $\langle i,n\rangle_r = \langle i,n\rangle \in R(r)$ — i.e., $r\langle i,n\rangle = \text{true}$. Certainly $\langle i,n\rangle \in \text{dom}(p)$, for if not then $\exists q \geq p$ with $q\langle i,n\rangle = \text{false}$ which is impossible by the above. But then $p\langle i,n\rangle = \text{true}$, since $\exists r \geq p$ with $r\langle i,n\rangle = \text{true}$. So, let $\varphi : \Delta I \times \Delta\omega \longrightarrow \Omega_{\neg\neg}$ be the characteristic map of $R \rightarrowtail \Delta I \times \Delta\omega$. Then we claim that the transpose

$$\bar{\varphi} : \Delta I \longrightarrow \Omega_{\neg\neg}^{\Delta\omega}$$

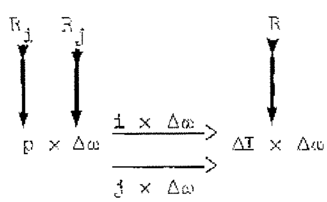
is monic. Well, if $i,j \in I$, the the composites

$$p \underset{j}{\overset{i}{\rightrightarrows}} \Delta I \xrightarrow{\ \bar{\varphi}\ } \Omega_{\neg\neg}^{\Delta\omega}$$

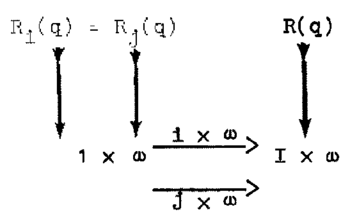
are equal iff the transposes

$$p \times \Delta\omega \underset{j \times \Delta\omega}{\overset{i \times \Delta\omega}{\rightrightarrows}} \Delta I \times \Delta\omega \longrightarrow \Omega_{\neg\neg}$$

are equal, iff in



the pullbacks R_i and R_j are equal. This is true iff $\forall q \geq p$ we have



But

$$R_i(q) = \{n | q\langle i,n\rangle = \text{true}\}$$

$$R_j(q) = \{n | q\langle j,n\rangle = \text{true}\} \quad ,$$

and these are not equal if $i \neq j$, since dom(p) is finite — i.e., if $i \neq j$, we can find an n such that neither $\langle i,n\rangle$ nor $\langle j,n\rangle$ is in dom(p). Then, however, $\exists q \geq p$ with $q\langle i,n\rangle$ = true and $q\langle j,n\rangle$ = false. Now we are done, for $\Omega_{\neg\neg}$ is a sheaf, thus the dense monomorphism $\Delta\omega \rightarrowtail \hat{\omega}$ induces an isomorphism

$$\Omega_{\neg\neg}^{\hat{\omega}} \xrightarrow{\sim} \Omega_{\neg\neg}^{\Delta\omega}$$

so we have a monomorphism

$$\Delta I \rightarrowtail \Omega_{\neg\neg}^{\hat{\omega}} \quad .$$

Applying the associated sheaf functor, which preserves monomorphisms, gives a monomorphism

$$\hat{I} \overset{\psi}{\rightarrowtail} \Omega_{\neg\neg}^{\hat{\omega}}$$

which is what we want, since $\omega = \hat{\omega}$ in $Sh_{\neg\neg}(\mathbb{P})$.

To finish the argument, consider, in $\underline{S}$, the maps

$$\omega \rightarrowtail 2^{\omega} \rightarrowtail I \quad .$$

Then $\text{Epi}((\omega,2^{\omega}) = 0 = \text{Epi}(2^{\omega},I)$ — the first by Cantor's diagonal argument and the second by hypothesis. Thus in $Sh_{\neg\neg}(\mathbb{P})$ we obtain

$$\omega = \hat{\omega} \rightarrowtail \widehat{2^{\omega}} \rightarrowtail \hat{I} \overset{\psi}{\rightarrowtail} 2^{\omega} \quad ,$$

and by Theorem 3, $\text{Epi}(\omega,\widehat{2^{\omega}}) = 0 = \text{Epi}(\widehat{2^{\omega}},\hat{I})$. But I has global sections, so ψ splits (by Booleanness). Let $q:2^{\omega} \twoheadrightarrow \hat{I}$ be a map such that $q\psi = \text{id}$. Since q is epic, it induces by composition a map

$$\text{Epi}(\widehat{2^{\omega}},2^{\omega}) \longrightarrow \text{Epi}(\widehat{2^{\omega}},\hat{I}) \quad .$$

Thus $\mathrm{Epi}(\widehat{2^{\omega}},2^{\omega}) = 0$ and $\widehat{2^{\omega}}$ is the X that negates CH.

Having now established the fact that CH does not follow from the seven axioms given for CS, it is only natural to ask what relation this result bears to that of Cohen for ZF. This, of course, amounts to asking for the relationship of CS to ZF. Very,very, briefly, the situation seems to be the following. Thinking of ZF sets as trees — i.e., as objects provided with a given (ε) relation, it becomes possible to pass back and forth between models of CS and models of set theory — in the usual sense. Starting with a model of CS, however, we can never hope to obtain in this way a model of ZF - CS, for example, is finitely axiomatisible. What is lacking is replacement. This is easy to formulate for CS, though still somewhat unclear for a general topos. We have not mentioned it here, since it plays no role in the argument — i.e., once assumed for CS, one will have to verify its presence in the various stages of the constructions, though as an axiom it will never be used in these constructions. In any case, once we add replacement to CS, the above process will yield an equivalence between models for CS and models for ZF. (This process has since been carried out by J.C. Cole [5], and W. Mitchell [6].) Thus, since CH is a categorical statement, its negation in one system will be equivalent with its negation in the other.

In closing, we might make a few remarks as to possible future uses for these sheaf theoretic methods — at least in so far as independence results in logic are concerned. Probably it is fair to say that though one can develop other logical constructions on topos that enable one to establish further classical independence results, for example AC can be handled in this way, it seems unlikely that these methods, using partially ordered sets, will yield many interesting new results in this area — largely because most of them have probably already been obtained by more standard techniques.

However, in this treatment we are able to deal with arbitrary categories of forcing conditions, not merely partially ordered sets, and this should prove to be a useful technique in model theory. For example, elementary theories themselves might prove to be interesting sites. Also, as indicated earlier, most topos are non-classical — in that Ω is not Boolean — and one can make use of this instead of discarding it by passing to $\neg\neg$ -sheaves. For example, it seems that the topological interpretation of intuitionism can be thought of simply as mathematics done in Sheaves (T) where T is a topological space. Many independence results in intuitionistic algebra and analysis should be provable by topos methods, though only the surface has been scratched to date.

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